Tropical Intersection Theory on Moduli Stack of Curve Coverings

Zhi Jin

DPMMS, Faculty of Mathematics
University of Cambridge

This dissertation is submitted for the degree of
Doctor of Philosophy

Churchill College September 2018

Tropical Intersection Theory on Moduli Stack of Curve Coverings

Zhi Jin

Abstract

We construct the moduli cone stack $\mathscr{M}_\eta^{\mathrm{trop}}$ of tropical étale covers (i.e., coverings of twisted tropical curves). We define the tropical intersection theory on $\mathscr{M}_\eta^{\mathrm{trop}}$ and show that the tropical intersection theory agrees with the intersection theory on the moduli stack $\overline{\mathscr{M}}_\eta$ of étale covers (i.e., coverings of twisted algebraic curves). We apply the tropical intersection theory on $\mathscr{M}_\eta^{\mathrm{trop}}$ to calculate the intersection numbers of Psi-classes on the moduli space $\overline{\mathscr{M}}_{g,n}$ of n-marked genus g curves. We also define the moduli stack $\mathscr{M}_\eta^{\log}$ of logarithmic étale covers and describe the tropicalization map from $\mathscr{M}_\eta^{\log}$ to the Artin fan of $\mathscr{M}_\eta^{\mathrm{trop}}$.

I would like to dedicate this thesis to my loving parents

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

Zhi Jin
September 2018

Acknowledgements

I am grateful for all the support I have received towards my degree of Doctor of Philosophy.

First I would like to thank my supervisor, Prof.Mark Gross, who has given remarkable advice, excellent guidance and endless patience to this research project. I have gained a great deal from his supervisions and instructions. I also benefit tremendously from the research group on Mirror symmetry he creates in Cambridge.

Conversations with my colleagues Lawrence Barrott, Mandy Cheung, Michael Kasa, Tyler Kelly, Ben Morley and Alan Thompson about the topics presented in this thesis are inspiring and helpful.

I would also like to thank the friends I have made through mathematics: Benjamin Barrett, Cangxiong Chen, Zexiang Chen, Si Cheng, Nina Friedrich, Michel van Garrel, Guolong Li, Christian Lund, Wicher Malten, Yi Man, Nils Prigge, Zhiyou Wu, Wenzhe Yang and Shuai Zhai. I am grateful to my friend Jingjun Han and Junliang Shen for their care and support.

Finally, I would like to thank my parents for their love and encouragement.

Table of contents

Nomenclature

$[n]$	The set $\{1,...,n\}$
$\Bbbk$	An algebraically closed field
$\underline{f}$	The underlying scheme morphism of the log morphism f
f^b	The log structure morphism induced by the log morphism f
M	the dual of the finitely generated free abelian group N
N	A finitely generated free abelian group

Introduction

We realize the tropical calculation of the intersection numbers of Psi-classes on the moduli stack $\overline{\mathscr{M}}_{g,n}$ for higher genus g. For the case $g = 0$, see [30] and [20]. For the intersection of stable Psi-classes, see [8]. Unlike the moduli stack $\overline{\mathscr{M}}_{0,n}$ of rational n-marked curves, the moduli stack $\overline{\mathscr{M}}_{g,n}$ of n-marked genus g in general does not embed into a toric variety nicely, i.e., we do not have an embedding of $\overline{\mathscr{M}}_{g,n}$ into some toric variety X such that there is natural correspondence between the cycles on $\overline{\mathscr{M}}_{g,n}$ and the cycles in X. Costello shows that intersection numbers of Psi-classes on the moduli stack $\overline{\mathscr{M}}_{g,n}$ can be calculated by intersecting cycles on the moduli stack $\overline{\mathscr{M}}_{\eta}$ of étale covers [12]. Moreover, there is a natural morphism $\overline{\mathscr{M}}_{\eta} \to \overline{\mathscr{M}}_{0,n}$ which allows us to transfer calculations from $\overline{\mathscr{M}}_{\eta}$ to $\overline{\mathscr{M}}_{0,n}$. Following this idea, we transfer the tropical intersection theory on $\mathscr{M}_{0,n}^{\text{trop}}$ to define the tropical intersection theory on the cone stack $\mathscr{M}_{\eta}^{\text{trop}}$ which agrees with the intersection theory on $\overline{\mathscr{M}}_{\eta}$. Then we can do the tropical calculation on $\mathscr{M}_{\eta}^{\text{trop}}$ to get the intersection numbers of $\overline{\mathscr{M}}_{g,n}$.

The motivation of this thesis is to tropicalize the calculation of Gromov-Witten invariants. For a general survey on Gromov-Witten invariants, see [15]. The development of mirror symmetry creates the interest in the Gromov-Witten invariants which count algebraic curves on a smooth projective variety. Mikhalkin first used tropical curves to give a count of algebraic curves on toric surfaces [33]. Then Nishinou and Siebert generalized the results to $g = 0$ for all complete toric varieties [36] where the tropicalization via logarithmic geometry in the Gross-Siebert program ([22], [24]) is used. It is also suspected that this approach should hold in general. The main issue is the "superabundance" phenomenon, i.e., the dimension of the moduli stack of algebraic curves does not match the dimension of the moduli space of tropical curves.

Costello shows that higher genus Gromov-Witten invariants of X can be expressed as genus zero invariants of symmetric products S^dX [12]. In particular there are certain finite maps between the moduli stack $\overline{\mathscr{M}}_{g,n}(X)$, $\overline{\mathscr{M}}_{\eta}(X)$ and $\overline{\mathscr{M}}_{0,n}(S^dX)$. We hope there are similar maps in the tropical geometry. Then we may be able to express higher genus tropical curve counting as rational curve counting and hence establish the higher genus equality of tropical curve counting and algebraic curve counting. In particular, we may be able to have a better understanding on the superabundance phenomenon, i.e., be able to identify which higher genus curves do not have corresponding algebraic curves. For result of genus 0 and 1 on this, see [42] and [39].

We then focus on the case $X = pt$ to avoid dealing with the tropicalization of $\overline{\mathscr{M}}_{0,n}(S^dX)$ which requires further understanding. The Gromov-Witten theory of a point has been known sice Kontsevich's proof [31] of Witten's conjecture [44]. So there is nothing new here in this respect.

Cavalieri, Chan, Ulirsch and Wise propose the stack-theoretic approach to construct the moduli space parametrizing the tropical curves [9]. This method can easily be generalized to define the moduli space $\mathscr{M}_\eta^{\mathrm{trop}}$ of tropical étale covers which is the tropicalization of twisted curves. Moreover, the tropical intersection theory on $\mathscr{M}_{0,n}$ defined by Kerber and Markwig [30] can be transferred easily to $\mathscr{M}_\eta^{\mathrm{trop}}$.

Gross and Siebert propose the tropicalization via logarithmic geometry in [23]. The process is made precise by Abramovich, Chen, Gross and Siebert in [2]. Gross gives a more systematic survey about how log geometry bridges tropical geometry and algebraic geometry in [21]. In the end, we compare the tropicalization via logarithmic geometry with the tropicalization via the cone stack using the methods proposed by Cavalieri, Chan, Ulirsch and Wise in [9].

The space $\mathscr{M}_{0,n}^{\mathrm{trop}}$ is the moduli cone stack of tropical curves of genus 0 with n marked points. A tropical curve of genus 0 with n marked points is a connected metric graph with n labeled unbounded edges such that any vertex has at least 3 edges. The metric means each bounded edge is assigned a number in $\mathbb{R}_+$. These curves are parametrized by the combinatorial structure of the underlying nonmetric graph and the length of each bounded edge. Notice the underlying nonmetric graph can have at most $n-3$ bounded edges. So the tropical moduli space $\mathscr{M}_{0,n}^{\mathrm{trop}}$ has the structure of a polyhedral complex, obtained by gluing several copies of the orthant $\mathbb{R}_{\geq 0}^{n-3}$—-one copy for each 3-valent graph with n unbounded edges.

The moduli cone stack $\mathscr{M}_{0,n}^{\mathrm{trop}}$ is exactly the tropicalization of $\overline{\mathscr{M}}_{0,n}$. For the tropicalization here, see [41] or [17]. For the tropicalization of $\overline{\mathscr{M}}_{0,n}$, see [35]. Indeed, given a rational stable curve with n marked points, we can construct the dual intersection graph of the stable curve. Recall stable curve has at worst nodal singularities. The dual intersection graph of the n-marked stable curve $(C, p_1, ..., p_n)$ has 1 vertex for each irreducible component. There is a bounded edge connecting two vertices if and only if the corresponding irreducible components intersect at a node. There are also n unbounded edges corresponding to n marked points. An unbounded edge k is attached to the vertex v if and only if the marked point p_k is on the irreducible component corresponding to v. A nonmetric graph G will specify the closed subvariety $\overline{\mathscr{M}}_G \subset \overline{\mathscr{M}}_{0,n}$ where the points of $\overline{\mathscr{M}}_G$ correspond to stable curves whose dual intersection graph is G directly or is G after some bounded edges are contracted. On the other hand, the tropicalization of $\overline{\mathscr{M}}_{0,n}$ is obtained by gluing several copies of the orthant $\mathbb{R}^{n-3}_{\geq 0}$—-one copy for each deepest degenerated stable curve. Notice that the dual intersection graphs of the deepest degenerated stable curves are precisely all 3-valent genus 0 graphs. The moduli cone stack $\mathscr{M}^{\text{trop}}_{0,n}$ is the tropicalization of $\overline{\mathscr{M}}_{0,n}$.

The calculation of the intersection theory on $\overline{\mathscr{M}}_{0,n}$ can be done via the tropical intersection theory on $\mathscr{M}^{\text{trop}}_{0,n}$. Indeed, Gibney and Maclagan show that the moduli stack $\overline{\mathscr{M}}_{0,n}$ embeds into a toric variety [18]. Moreover, the embedding respects the intersection theory, i.e., the Chow cohomology classes in which we are interested come naturally from the pullback of this embedding. Then it is possible to define the tropical intersection theory on $\mathscr{M}^{\text{trop}}_{0,n}$ by applying the concepts suggested in [34] and developed in detail in [7]. For tropical calculation of intersection of psi-classes on $\mathscr{M}^{\text{trop}}_{0,n}$, see [32].

The idea of tropical intersection theory is to take advantage of the property of toric varieties. For a complete toric variety X of dimension n, we have

$$\mathscr{D}_k : A^k X \xrightarrow{\cong} \mathrm{Hom}(A_k X, \mathbb{Z}),$$

where $A^k X$ is the codimension k Chow cohomology group and $A_k X$ is the dimension k Chow group [16, Proposition 2.4]. Recall each $(n-k)$-dimensional cone σ in the tropicalization of X represents a closed subvariety D_σ of dimension k in X. Moreover, the Chow group $A_k X$ is generated by these closed subvarieties. A *tropical fan of dimension k* is a collection of dimension k cones of the fan for X with a weight function assigning to each cone a weight in $\mathbb{Z}$ satisfying certain balancing condition. Then the group $A^{n-k}X$ is isomorphic to the group of *tropical fans of dimension k* via the map

$$c \mapsto Y_c$$

where c is an element in $A^{n-k}X$; the weight function of the k-dimensional tropical fan Y_c assigns the intersection number $c.[D_\sigma]$ to each k-dimensional cone σ. The balancing condition comes from the relations between D_σ in $A_{n-k}X$.

Furthermore, each continuous piecewise linear function on the tropicalization of the n-dimensional toric variety X gives a Cartier divisor on X, i.e., an element in $A^1 X$. Indeed, a piecewise linear function is linear on each n-dimensional cone. So specifying a piecewise linear funtion is the same as specifying an element m_σ in the dual space for each n-dimensional cone σ, which is the same as specifying a rational function on the each open set $V(\sigma)$. These open sets $V(\sigma)$ form a cover of X. Moreover, continuity of the piecewise linear function translates to the condition that on the intersection of two such open sets, the quotient of the two rational functions is a regular function. So we have a Cartier divisor. For more details, see [14] and [16].

Now we can intersect a continuous piecewise linear function with a k-dimensional tropical fan just like intersecting a Cartier divisor with an element in $A^{n-k}X$. Moreover, the calculation is tropical, i.e., only tropical data (weight, faces, piecewise linear function...) are needed in the calculation. The result will be a $(k-1)$-dimensional tropical fan.

Since Gibney and Maclagan show that $\overline{\mathscr{M}}_{0,n}$ embeds nicely into some complete toric variety X and $\overline{\mathscr{M}}_{0,n}^{\mathrm{trop}}$ is a subfan of the tropicalization of X [18], all these tropical calculations generalize naturally to $\overline{\mathscr{M}}_{0,n}^{\mathrm{trop}}$. To intersect Psi-classes ψ_i or boundary divisors on $\overline{\mathscr{M}}_{0,n}$, it suffices to find the corresponding piecewise linear function and the tropical fan associated to the fundamental class $[\overline{\mathscr{M}}_{0,n}]$. See [30].

Because $\overline{\mathscr{M}}_{g,n}$ does not embed nicely into toric varieties, it is difficult to directly generalize the above and to use only tropical data to intersect Psi-classes. Costello shows that the descendent genus g Gromov-Witten invariants (i.e., intersection of Psi-classes on $\overline{\mathscr{M}}_{g,n}$) can be expressed as genus 0 Gromov-Witten invariants [12]. We will describe the idea below.

An étale cover is a degree d étale morphism $\mathscr{C}' \to \mathscr{C}$ of twisted balanced nodal curves satisfying some other conditions. The moduli stack $\overline{\mathscr{M}}_\eta$ parametrizes such étale covers with certain markings on $\mathscr{C}'$ and $\mathscr{C}$ where the genus $g(\mathscr{C}) = 0$. The genus of a twisted curve is that of its coarse moduli space. Here η is some label remembering the genera of $\mathscr{C}'$ and $\mathscr{C}$, the stack (orbifold) structure at the marked points and so on. These stacks are studied in [4] and [5] where they are related to more classical moduli stacks of twisted and ordinary stable curves.

There is a map $p : \overline{\mathscr{M}}_\eta \to \overline{\mathscr{M}}_{g,n}$ for g and n determined by η, defined by taking the coarse moduli space C' of $\mathscr{C}'$ and forgetting some marked points. If η is chosen carefully, the map p is finite of degree $k \in \mathbb{Q}^\times$, i.e.,

$$p_*[\overline{\mathscr{M}}_\eta] = k[\overline{\mathscr{M}}_{g,n}],$$

where k can be calculated combinatorically based on η. Then the pull back $p^*\psi_i$ of the Psi-class ψ_i on $\overline{\mathscr{M}}_{g,n}$ can be expressed in terms of Psi-classes and boundary divisors of $\overline{\mathscr{M}}_\eta$.

This translates the descendent Gromov-Witten invariants on $\overline{\mathscr{M}}_{g,n}$ into intersection theory on $\overline{\mathscr{M}}_{\eta}$.

On the other hand, there is a map $rt : \overline{\mathscr{M}}_{\eta} \to \overline{\mathscr{M}}_{0,m}$ defined by taking the coarse moduli space C of the target twisted curve $\mathscr{C}$. The map is proper and flat. This map factorizes as

$$\overline{\mathscr{M}}_{\eta} \xrightarrow{t} \overline{\mathscr{M}}_{\eta_t} \xrightarrow{r} \overline{\mathscr{M}}_{0,m},$$

where $\overline{\mathscr{M}}_{\eta_t}$ is the moduli stack of twisted curves. Here η_t is the label for twisted curves. The first map t is étale and the ramifications of the second map r at divisors are easily calculated by looking at the local picture. So we can pull back divisors from $\overline{\mathscr{M}}_{0,m}$ to $\overline{\mathscr{M}}_{\eta}$.

To summarize, there is a correspondence $\overline{\mathscr{M}}_{0,m} \leftarrow \overline{\mathscr{M}}_{\eta} \to \overline{\mathscr{M}}_{g,n}$ which is finite over both $\overline{\mathscr{M}}_{0,m}$ and $\overline{\mathscr{M}}_{g,n}$. The most important thing is that the étaleness of t enables us to describe the map $\overline{\mathscr{M}}_{\eta} \to \overline{\mathscr{M}}_{\eta_t}$ tropically. Moreover, the effect of the map r can be dealt with by simply adding a factor.

A *twisted tropical curve* is a metric graph with some extra data assigning a positive integer to each edge. Recall the underlying nonmetric graph of a tropical curve can be viewed as the dual intersection graph of some stable curves. The underlying nonmetric twisted graph of a twisted tropical curve can be viewed as the dual intersection graph of some twisted curve with the extra data recording the stack (orbifold) structure at the nodes and the marked points.

A *tropical étale cover* is a morphism $\Gamma_s \to \Gamma_t$ of metric twisted tropical curves satisfying certain conditions. The conditions come from the fact that the underlying nonmetric graph of a tropical étale cover is the dual intersection graph of some étale cover. The metric on the source twisted tropical curve Γ_s is always induced from the metric on the target twisted tropical curve by the morphism. So we usually specify a tropical étale cover by the morphism $G_s \to G_t$ of the underlying nonmetric twisted graphs and the metric on G_t.

The moduli space $\mathscr{M}_{\eta}^{\mathrm{trop}}$ parametrizing tropical étale covers is constructed as a cone stack following the process constructing $\mathscr{M}_{g,n}^{\mathrm{trop}}$ in [9]. For the tropicalization of $\mathscr{M}_{\eta}^{\mathrm{trop}}$ via Berkovich analytification, see [10].

In [9], the moduli space $\mathscr{M}_{g,n}^{\mathrm{trop}}$ is treated as a stack instead of as a set to make the tropical forgetful map $\pi_{g,n+1} : \mathscr{M}_{g,n+1}^{\mathrm{trop}} \to \mathscr{M}_{g,n}^{\mathrm{trop}}$ become the universal curve. Following the idea, a stack-theoretic approach is taken: we define the moduli space $\mathscr{M}_{\eta}^{\mathrm{trop}}$ as a tropical moduli functor

$$\mathscr{M}_{\eta}^{\mathrm{trop}} : \mathbf{RPC}_{\mathbb{Z}} \to \mathbf{Groupoids}$$

that associates to a rational polyhedral cone σ the groupoid of tropical étale covers Γ with edge lengths taking values in the dual monoid S_σ to σ. Taking lengths in S_σ allows keeping track of higher-dimensional deformation parameters of multi-parameter degenerations.

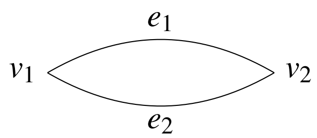
The functor $\mathscr{M}_\eta^{\mathrm{trop}}$ is then proved to be a cone stack. The following analogy with algebraic geometry is given in [9]. We copy it here to help understanding:

algebraic geometry	tropical geometry
rings	monoids
affine schemes	cones
schemes	cone complexes
algebraic spaces	cone spaces
algebraic stacks	cone stacks.

Like in algebraic geometry, the stack structure of $\mathscr{M}_{g,n}^{\mathrm{trop}}$ comes from the automorphisms of tropical curves. For example, given two edges e_1 and e_2 connecting two same vertices, the map which maps $e_1 \mapsto e_2, e_2 \mapsto e_1$ and is the identity elsewhere is an automorphism of the tropical curve. The stack structure of $\mathscr{M}_\eta^{\mathrm{trop}}$ comes from the automorphisms of tropical étale covers as well.

Notice that the cone stack $\mathscr{M}_{0,n}^{\mathrm{trop}}$ is actually a cone complex as expected. Moreover, the cone stack definition matches the previous definition for $\mathscr{M}_{0,n}^{\mathrm{trop}}$.

Assume again we have two edges e_1 and e_2 connecting the same pair of vertices (v_1, v_2).



Then the tropical curves with such underlying nonmetric graph should be parametrized by $\mathbb{R}_{\geq 0}^2$ with entries being the lengths of e_1 and e_2. Hence we have to identify (x,y) with (y,x) because the tropical curves are isomorphic. This is another reason why tropical intersection theory on $\mathscr{M}_{g,n}^{\mathrm{trop}}$ is difficult to realize.

There is a map $tc : \mathscr{M}_\eta^{\mathrm{trop}} \to \mathscr{M}_{0,n}^{\mathrm{trop}}$ mapping a tropical étale cover to the coarse tropical curve of its target twisted tropical curve, i.e., forgetting the extra data of the target twisted tropical curve. We focus on the case where target twisted curve is of genus 0. Here genus means the genus assigned to each vertex plus the genus of the underlying nonmetric graph. Then any automorphism will be the identity on the target twisted curve. Recall that tropical étale covers are parametrized by the lengths on the target twisted curves. All the stack structure on $\mathscr{M}_\eta^{\mathrm{trop}}$ is induced by the identity automorphism. (This does not mean there is no stack structure. The situation is similar to the classifying stack BG in algebraic geometry.) The map tc is precisely forgetting the stack structure locally.

For each cone σ of $\mathscr{M}_{0,n}^{\mathrm{trop}}$, there are several cone stacks lying over σ in $\mathscr{M}_{\eta}^{\mathrm{trop}}$ via tc. All these cone stacks have stack structure induced by identity automorphisms. The coarse moduli cones of these cone stacks are precisely σ. So the tropical intersection theory on $\mathscr{M}_{0,n}^{\mathrm{trop}}$ can be transferred to $\mathscr{M}_{\eta}^{\mathrm{trop}}$ easily. For example, piecewise linear functions on $\mathscr{M}_{\eta}^{\mathrm{trop}}$ can be defined as piecewise linear on the coarse moduli cones of each cone stack. When defining tropical fans and assigning weights to each cone stack, we simply add an extra factor accounting for automorphisms. In this way, we have a tropical intersection theory on $\mathscr{M}_{\eta}^{\mathrm{trop}}$ which agrees with the intersection theory on $\overline{\mathscr{M}}_{\eta}$.

As an application, we pull back Psi-classes of $\overline{\mathscr{M}}_{g,n}$ to $\overline{\mathscr{M}}_{\eta}$ via the formula provided in [12]. Then we calculate the intersections tropically on $\mathscr{M}_{\eta}^{\mathrm{trop}}$ to get the intersection numbers.

In the final chapter we construct the moduli stack $\mathscr{M}_{\eta}^{\log}$ of basic log étale covers as a logarithmic stack following the method in [23]. The logarithmic stack is in the sense of [28] and [37]. We will use "log" instead of "logarithmic" for short. Then we show there is a smooth strict tropicalization morphism $\mathscr{M}_{\eta}^{\log} \to \mathfrak{a}^*\mathscr{M}_{\eta}^{\mathrm{trop}}$ where $\mathfrak{a}^*\mathscr{M}_{\eta}^{\mathrm{trop}}$ is the Artin fan of $\mathscr{M}_{\eta}^{\mathrm{trop}}$ following the process in [9].

Logarithmic geometry developed in [27] and [29] serves as a connection between algebraic geometry and tropical geometry. In particular, log geometry describes multi-parameter degenerations of algebraic curves elegantly.

The main concept in constructing the log stack $\mathscr{M}_{\eta}^{\log}$ are the *basic log étale covers*. For similar concept for log stable curves, see [23] and [28]. A *log étale cover* is a log étale morphism of log stable curves. The reason for restricting to basic log étale covers is that the moduli stack of all log étale covers is too large because it allows arbitrary log structures on the family and the base. The notion of basicness selects a universal choice of log structure on the base. Basicness only depends on the underlying morphism of stable curves. The relation to tropical geometry comes from pulling back to the standard log point (Example 4.1.10).

The process of tropicalization is described as a functor on the fibers of the moduli stack $\mathscr{M}_{\eta}^{\log}$ of log étale covers over log points, i.e., the scheme $\mathrm{Spec}\Bbbk$ with log structure.

Let S be the log point $(\mathrm{Spec}\Bbbk, k^* \oplus Q)$, where Q is a monoid. A log smooth curve $X \to S$ can be thought of as a marked stable curve $\underline{X}$, where each geometric point x of $\underline{X}$ is endowed with a monoid $\overline{M}_{\underline{X},x}$. The structure of such monoids is given in [28, Theorem 1.3]:

- At general points: $\overline{M}_{\underline{X},x} \simeq Q$;
- At marked points: $\overline{M}_{\underline{X},x} \simeq Q \oplus \mathbb{N}$;

- At a node q: $\overline{M}_{\underline{X},x} \simeq Q \oplus_{\mathbb{N}} \mathbb{N}^2$ where $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ is determined by $1 \mapsto \alpha_q \in Q$ and $1 \mapsto (1,1)$.

One may think of α_q as the smoothing parameter at the corresponding node. A *log stable curve* is a log smooth curve whose underlying curve is stable. For log étale covers, we have the general points (marked points, nodes respectively) sit over general points (marked points, nodes respectively).

The tropical étale cover associated to a log étale cover $C_s \to C_t \to S$ is the dual intersection graph associated to the underlying morphism $\underline{C_s} \to \underline{C_t}$ of stable curves, together with the twisted degree given by the ramification of the underlying morphism and the metric at each node given by the smoothing parameter α_q. This association defines the tropicalization which is a natural functor:

$$\mathrm{trop} : \mathscr{M}_\eta^{\log}(S) \to \mathscr{M}_\eta^{\mathrm{trop}}(\sigma_S)$$

where $\sigma_S = \mathrm{Hom}(Q, \mathbb{R}_{\geq 0})$. The functor trop specializes to the tropicalization map as in [23, Appendix B].

The theory of *Artin fans* was developed in [6] and [3] which serves as an interface between log geometry and tropical geometry. Every cone stack can be lifted to a log algebraic stack which is the Artin fan associated to the cone stack. The lifting functor is denoted by $\mathfrak{a}^*$ which defines an equivalence between the 2-category of cone stacks and the 2-category of Artin fans [9, Theorem 6.15]. For a rational polyhedral cone σ, the Artin fan associated to σ is

$$\mathscr{A}_\sigma := ([V_\sigma / T], L)$$

where V_σ is the toric variety $\mathrm{Spec}\,\mathbb{k}[\sigma^\vee \cap M]$ associated to the polyhedral cone σ; the group T is the dense torus in the toric variety V_σ; the log structure L is induced by the divisorial log structure on V_σ associated to $V_\sigma \backslash T$.

The lifting of the cone stack $\mathscr{M}_\eta^{\mathrm{trop}}$ is the log algebraic stack $\mathfrak{a}^*\mathscr{M}_\eta^{\mathrm{trop}}$. The log algebraic stack $\mathfrak{a}^*\mathscr{M}_\eta^{\mathrm{trop}}$ may be reinterpreted as a moduli functor

$$\mathfrak{a}^*\mathscr{M}_\eta^{\mathrm{trop}} : \mathbf{LSch} \to \mathbf{Groupoids}$$

that associates to a log scheme S the groupoid of *families of tropical étale covers* over S, i.e., of collections (Γ_x) of tropical étale covers indexed by the geometric points x of S with a generalized edge length with values in the characteristic monoid $\overline{M}_{S,x}$ that are compatible under specialization. The natural functor $\mathrm{trop} : \mathscr{M}_\eta^{\log}(S) \to \mathscr{M}_\eta^{\mathrm{trop}}(\sigma_S)$ induces the morphism

$$\mathscr{M}_\eta^{\log}(S) \to \mathfrak{a}^* \mathscr{M}_\eta^{\mathrm{trop}}(S).$$

We then show the morphism $\mathscr{M}_\eta^{\log} \to \mathfrak{a}^* \mathscr{M}_\eta^{\mathrm{trop}}$ of log algebraic stack is smooth and strict.

Further work would involve generalizing the result to X of higher dimension. In particular, we need a tropical explanation of the stack $S^d X$ and have a tropical intersection theory on the moduli space $\mathscr{M}_{0,n}^{\mathrm{trop}}(S^d X)$ which parametrizes tropical curves with a map to the tropicalization of $S^d X$. Future work in another direction would involve comparing $\mathscr{M}_\eta^{\log}$ and $\overline{\mathscr{M}}_\eta$.

Chapter 1

Intersection theory on $\overline{\mathscr{M}}_{0,n}$ via tropical geometry

1.1 Tropical intersection theory

In this section we will discuss fans and toric varieties. In particular, we are interested in the intersection theory on fans. This has been studied in [14] and [7].

Let N be a finitely-generated free abelian group, i.e. a group isomorphic to $\mathbb{Z}^r$, and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ be the associated real vector space containing N as a lattice. We denote the dual lattice in the dual vector space by $M \subset M_{\mathbb{R}}$.

A *cone* is a subset $\sigma \subset N_{\mathbb{R}}$ that can be described by finitely many linear integral equalities and inequalities, i.e. a set of the form

$$\sigma = \{x \in N_{\mathbb{R}} | f_1(x) = 0, ..., f_s(x) = 0, f_{s+1}(x) \geq 0, ..., f_N(x) \geq 0\}$$

for some linear forms $f_1, ..., f_N \in M$. We denote by V_σ the smallest linear subspace of $N_{\mathbb{R}}$ containing σ and by Λ_σ the lattice $V_\sigma \cap \Lambda$. The dimension of σ is the dimension of V_σ.

A cone of dimension r is *simplicial* if it can be generated by positive linear combinations of r vectors.

A *fan* X is a finite collection of cones in $N_{\mathbb{R}}$ such that intersections of cones in X belong to X and every cone $\sigma \in X$ is the disjoint union

$$\sigma = \bigsqcup_{\tau \in X, \tau \subset \sigma} \mathring{\tau},$$

where $\mathring{\tau}$ denotes the interior of τ in V_τ.

A fan is *simplicial* if all of its cones are simplicial.

We will denote the set of all k-dimensional cones of X by $X^{(k)}$. The *dimension* of X is the maximum of the dimensions of the cones in X. The fan X is called *pure-dimensional* if each cone is contained in some cone of dimension $\dim X$ in X. The union of all cones in X will be denoted $|X| \subset N_{\mathbb{R}}$. If X is pure-dimensional then cones in $X^{(\dim X)}$ are called *facets* of X. A fan X is *complete* if $|X| = N_{\mathbb{R}}$.

If $\tau \subset \sigma$ in X then τ is called a *face* of σ which we denote as $\tau \leq \sigma$. If $\tau \leq \sigma$ and $\tau \neq \sigma$ then τ is a *proper* face of σ, which we denote as $\tau < \sigma$.

Let $\tau < \sigma$ be cones of some fan X with $\dim \tau = \dim \sigma - 1$. This implies $\Lambda_\sigma / \Lambda_\tau \cong \mathbb{Z}$. Let $u_\sigma \in \sigma \cap \Lambda_\sigma$ be a vector whose class $u_{\sigma/\tau} := [u_\sigma] \in \Lambda_\sigma/\Lambda_\tau$ generates Λ_σ / V_τ. Note this class $u_{\sigma/\tau}$ does not depend on the choice of u_σ. We call $u_{\sigma/\tau}$ the *primitive normal vector* of σ relative to τ.

Definition 1.1.1. [7, Definition 2.5] A *weighted fan* (X, ω_X) of dimension k is a fan X of pure dimension k, together with a map $\omega_X : X^{(k)} \to \mathbb{Z}$. The number $\omega_X(\sigma)$ is called the *weight* of the facet $\sigma \in X^{(k)}$.

We write $\omega(\sigma)$ for $\omega_X(\sigma)$ when there is no ambiguity. Furthermore, we omit the weight function ω_X when ω_X is clear from the context.

Definition 1.1.2. [7, Definition 2.5] A *tropical fan* of dimension k is a weighted fan (X, ω_X) of dimension k satisfying the following *balancing condition* for every $\tau \in X^{(k-1)}$:

$$\sum_{\sigma : \tau < \sigma} \omega_X(\sigma) \cdot u_{\sigma/\tau} = 0 \in N_{\mathbb{R}}/V_\tau.$$

This is the Minkowski weight in [16]. The balancing condition is equivalent to that for any representation $\tilde{u}_{\sigma/\tau} \in N$ of $u_{\sigma/\tau}$,

$$\sum_{\sigma : \tau < \sigma} \omega_X(\sigma) \cdot \tilde{u}_{\sigma/\tau} \in V_\tau.$$

A more general definition would allow subdivisions. We denote the group of tropical fans by $\mathrm{TF}^*(X)$ with the addition being addition of weights for fans of the same dimension and formal addition for fans of different dimension. Let $\mathrm{TF}^k(X)$ be the group of dimension k tropical fans of X. Then the group $\mathrm{TF}^k(X)$ is precisely the group of Minkowski weights of codimension $n-k$ in [16].

The weighted fans play the role of cycles while the tropical fans play the role of cocyles in the intersection theory. Indeed, on one hand, the weights can be viewed as coefficients in front of the cycles. On the other hand, the weights can be viewed as the value of a cocycle on different cycles. While there is no restriction on the coefficients in front of cycles, the value of a cocycle on different cycles has to satisfy some conditions which translates exactly into the balancing condition of tropical fans. There is a toric variety $V(X)$ associated to each fan X. Moreover, the intersection theory on $V(X)$ gives us the intersection theory on X. We will explain further.

A *monoid* is a commutative semi-group with a unit. We denote the operation by $+$ and the unit by 0. A morphism of monoids $f : P \to Q$ is a map satisfying $f(0) = 0$ and $f(p+p') = f(p) + f(p')$ for any $p, p' \in P$. The *monoid ring* $\Bbbk[P]$ associated to a monoid P is the $\Bbbk$ vector space

$$\Bbbk[P] := \bigoplus_{p \in P} \Bbbk z^p,$$

with $\Bbbk$-bilinear multiplication determined by

$$z^p \cdot z^{p'} := z^{p+p'}.$$

Let X be a fan contained in $N_{\mathbb{R}}$ and σ a cone in X. The *dual cone* $\sigma^\vee \subset M_{\mathbb{R}}$ is

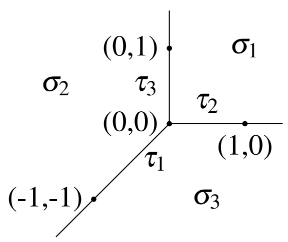
$$\sigma^\vee := \{m \in M_{\mathbb{R}} \mid \langle n, m\rangle \geq 0, \forall n \in \sigma\}.$$

The variety $V(\sigma)$ is defined as $\mathrm{Spec}\Bbbk[\sigma^\vee \cap M]$, where $\sigma^\vee \cap M$ is a monoid. If $\tau \in X$ is a face of σ, then $V(\tau)$ is an open subset of $V(\sigma)$. So we have an *associated variety* $V(X)$ with an open cover $V(X) = \bigcup_{\tau \in X} V(\tau)$, where we identify $V(\tau) \subset V(\sigma_1)$ with $V(\tau) \subset V(\sigma_2)$ whenever τ is a face of both σ_1 and σ_2.

The toric variety $V(X)$ is complete if and only if X is complete.

Example 1.1.3. Let N be $\mathbb{Z}^2$ and X be the collection of $\sigma_1, \sigma_2, \sigma_3$ and their proper faces where

$$\begin{aligned} \sigma_1 &= \mathbb{R}_{\geq 0}(0,1) + \mathbb{R}_{\geq 0}(1,0) \\ \sigma_2 &= \mathbb{R}_{\geq 0}(-1,-1) + \mathbb{R}_{\geq 0}(0,1) \\ \sigma_3 &= \mathbb{R}_{\geq 0}(-1,-1) + \mathbb{R}_{\geq 0}(1,0). \end{aligned}$$



We claim $V(X)$ is $\mathbb{P}^2$. Indeed, $X = \{0, \tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2, \sigma_3\}$, so the affine toric varieties are

$$\begin{aligned}
&V(0) = \operatorname{Spec}\Bbbk[z^{(1,0)}, z^{(0,1)}, z^{(-1,0)}, z^{(0,-1)}], && V(\tau_1) = \operatorname{Spec}\Bbbk[z^{(-1,1)}, z^{(-1,0)}, z^{(1,-1)}],\\
&V(\tau_2) = \operatorname{Spec}\Bbbk[z^{(1,0)}, z^{(0,1)}, z^{(0,-1)}], && V(\tau_3) = \operatorname{Spec}\Bbbk[z^{(1,0)}, z^{(0,1)}, z^{(-1,0)}],\\
&V(\sigma_1) = \operatorname{Spec}\Bbbk[z^{(1,0)}, z^{(0,1)}], && V(\sigma_2) = \operatorname{Spec}\Bbbk[z^{(-1,1)}, z^{(-1,0)}],\\
&V(\sigma_3) = \operatorname{Spec}\Bbbk[z^{(1,-1)}, z^{(0,-1)}],
\end{aligned}$$

Notice that actually $V(X)$ can be covered by $V(\sigma_i), i = 1,2,3$. And if we write for example $V(\sigma_1) = \operatorname{Spec}\Bbbk[x,y]$ with $x = z^{(1,0)}$ and $y = z^{(0,1)}$, then $V(\tau_3)$ will be $\{x \neq 0\}$ and $V(\tau_2)$ will be $\{y \neq 0\}$. Similar proposition holds for all $V(\sigma_i)$. Moreover, the intersection of $V(\sigma_1)$ and $V(\sigma_3)$ is $V(\tau_2)$ which is just $\{y \neq 0\}$ in $V(\sigma_1)$. Consider $\mathbb{P}^2 = \operatorname{Proj}\Bbbk[x_1, x_2, x_3]$ covered by $U_i := \{x_i \neq 0\}, i = 1,2,3$. Identifying U_i with $V(\sigma_i)$, $U_i \cap U_j$ with $V(\tau_k), (k \neq i$ or $j)$ and $U_1 \cap U_2 \cap U_3$ with $V(0)$, we get the isomorphism $V(X) \cong \mathbb{P}^2$.

The *toric stratum* D_σ corresponding to the cone $\sigma \in X$ is the closure of the orbit corresponding to σ:

$$D_\sigma := \overline{X_\sigma \setminus (\bigcup_{\tau:\tau<\sigma} X_\tau)}.$$

The functor $\sigma \mapsto D_\sigma$ is inclusion-reversing, i.e. if τ is a face of σ, then D_σ sits in D_τ as a closed subset. Moreover, $\operatorname{codim} D_\sigma = \dim \sigma$. In particular, the stratum D_0 is the whole toric variety $V(X)$.

A *Cartier divisor* on a fan X is a continuous piecewise linear function f on $|X|$ such that when restricted to each cone $\sigma \in X$ the function

$$f(x) = \langle m_\sigma, x\rangle$$

for some $m_\sigma \in M$. The $\mathbb{Q}$- or $\mathbb{R}$-*Cartier divisor* is defined if $m_\sigma \in M_\mathbb{Q} := M \otimes_\mathbb{Z} \mathbb{Q}$ or $m_\sigma \in M_\mathbb{R}$.

Assume N has rank r and X is pure-dimensional of dimension r. A Cartier divisor f on X gives a scheme theoretic Cartier divisor on $V(X)$. Indeed, the variety $V(X)$ has an affine open cover $V(X) = \bigcup_{\sigma \in X^{(r)}} V(\sigma)$. On each affine variety $V(\sigma)$ where $\sigma \in X^{(r)}$, the function f is given by a unique $m_\sigma \in M$. Remember z^{m_σ} is a regular function on $V(\sigma)$. The intersection of two open set $V(\sigma_1) \cap V(\sigma_2)$ is $V(\tau)$ where τ is the common face of σ_1 and σ_2. On $V(\tau)$, the function $z^{m_{\sigma_1} - m_{\sigma_2}}$ is invertible because both $m_{\sigma_1} - m_{\sigma_2}$ and $m_{\sigma_2} - m_{\sigma_1}$ lie in $\tau^\vee \cap M$. The

Cartier divisor D_f on $V(X)$ associated to f is represented by

$$\{(V(\sigma), z^{m_\sigma})|\sigma \in X^{(r)}\}.$$

The map $f \mapsto D_f$ behaves nicely with respect to the addition. For example, $D_{f+g} = D_f + D_g$, so $D_{af} = aD_f$ for $a \in \mathbb{Z}$. If f and g differ by a linear function $u \in M$ on $|X|$, we say $f \sim g$ are linear equivalent. Then D_{f-g} will be the principal Cartier divisor D_u. The *Cartier class group $CaCl(X)$* is the group of Cartier divisor modulo linear equivalence.

Now we have a morphism from the Cartier class group $CaCl(X)$ to the Chow cohomology $A^1(V(X))$ via a map to the Cartier class group on $V(X)$.

Proposition 1.1.4. *Assume X is a complete fan of dimension r and $V(X)$ is nonsingular. The morphism $i : CaCl(X) \to A^1V(X)$ given by*

$$f \mapsto \{(V(\sigma), z^{m_\sigma})|\sigma \in X^{(r)}\}$$

composed with the homomorphism $CaCl(V(X)) \to A^1V(X)$ is a group isomorphism.

Proof. First $CaCl(V(X)) \to A^1V(X)$ is an isomorphism. Indeed, the morphism $CaCl(V(X)) \to \mathrm{Pic}V(X)$ is isomorphism because $V(X)$ is integral [25, Proposition 6.15]. The morphism $\mathrm{Pic}V(X) \to A^1V(X)$ is isomorphism because X is complete [16, Corollary 3.4].

If $f \sim g$, then $D_f - D_g$ is the principle Cartier divisor z^{f-g} on $V(X)$. So the morphism $CaCl(X) \to CaCl(V(X))$ is well-defined. Hence the morphism i is well-defined.

Let D_τ be the toric stratum corresponding to the cone $\tau \in X^{(1)}$. Then $A^1(V(X))$ is generated by all D_τ [14, Section 5.1].

Let u_τ be the generator of $\tau \cap N$. Define f_τ be the continuous piecewise linear function obtained by extending $u_\tau \mapsto 1, u_{\tau'} \mapsto 0$ where τ' runs over $X^{(1)}$ except τ. This is possible because $V(X)$ is nonsingular so each cone of X is generated by part of a basis for the lattice [14, Section 2.1]. Then the Weil divisor corresponds to f_τ on $V(X)$ is D_τ. So the morphism is surjective.

Now suppose D_f is principal, i.e., a Cartier divisor on $V(X)$ given by the global function F. The Cartier divisor D_f is T-invariant on $V(X)$, where T is the toric action. The rational function F on $V(X)$ which is T-invariant has to be $\lambda \cdot z^{m_f}$ for some $m_f \in M$ and some number $\lambda \in \Bbbk$ because F is either identically zero or an invertible function on the big torus. So f is given by $m_f \in M$ on $|X|$. So the morphism is injective. □

Let X be a fan in N of pure dimension $r = \operatorname{rank} N$. For an element c in the Chow cohomology $A^*V(X)$, we can define a formal sum Y of weighted fans in X

$$c \mapsto Y_c := (Y_0, \omega_0) + (Y_1, \omega_1) + ... + (Y_r, \omega_r),$$

where (Y_i, ω_i) is a weighted fan of dimension i. Define the weight

$$\omega_i(\sigma) := c(D_\sigma), \sigma \in X^{(i)}.$$

Recall the toric stratum D_σ is of dimension $r - \dim\sigma$. So if c is in $A^k(V(X))$, then Y is a weighted fan of dimension $r-k$.

Proposition 1.1.5. *Each weighted fan in the sum Y_c is a tropical fan, i.e., (Y_i, ω_i) satisfies the balancing condition.*

Proof. To show the balancing condition for (Y_i, ω_i), it suffices to consider the component $c_{r-i} \in A^{r-i}V(X)$ of c.

Take $\tau \in X^{(i-1)}$ and $m \in \tau^\perp$ which is an element in M with value 0 on τ. Then on $V(X)$ the intersection of the principal Cartier divisor $div(z^m)$ and D_τ is

$$\sum_{\sigma:\sigma\in X^{(i)},\tau<\sigma} m(u_{\sigma/\tau}) D_\sigma,$$

which is zero in Chow cohomology $A^*V(X)$. Because $m \in \tau^\perp$, the value $m(u_{\tau/\sigma})$ is well defined. Moreover, we have

$$\begin{aligned}
&m\Big(\sum_{\sigma:\sigma\in X^{(i)},\tau<\sigma} u_{\sigma/\tau}\cdot\omega_i(\sigma)\Big)\\
=&\sum_{\sigma:\sigma\in X^{(i)},\tau<\sigma} m(u_{\sigma/\tau})\cdot\omega_i(\sigma)\\
=&\sum_{\sigma:\sigma\in X^{(i)},\tau<\sigma} m(u_{\sigma/\tau})\cdot c_{n-i}(D_\sigma)\\
=&c_{n-i}\Big(\sum_{\sigma:\sigma\in X^{(i)},\tau<\sigma} m(u_{\sigma/\tau})D_\sigma\Big)\\
=&c_{n-i}(div(z^m).D_\tau)\\
=&0.
\end{aligned}$$

This holds for all $m \in \tau^{\perp}$, hence the balancing condition

$$\sum_{\sigma:\sigma\in X^{(i)},\tau<\sigma} u_{\sigma/\tau} \cdot \omega_i(\sigma) = 0 \in N_{\mathbb{R}}/V_\tau$$

holds. $\square$

The construction $c \mapsto Y_c$ actually maps an element in Chow cohomology to a tropical fan and preserves the addition. So we have a group morphism

$$\mathscr{D}_X : A^k(V(X)) \to \mathrm{TF}^{n-k}(X).$$

Proposition 1.1.6. *[16, Theorem 3.1] If X is a complete fan, then the morphism $\mathscr{D}_X$ is an isomorphism.*

The Chow cohomology has a ring structure. In particular, the group $A^1(V(X))$ acts on $A^*(V(X))$. The Cartier class group $CaCl(V(X))$ acts on $A^*(V(X))$ if we view an element in $CaCl(V(X))$ as an element in $A^1(V(X))$. We wish to build similar action on the tropical world, i.e., the diagram

$$\begin{tikzcd} CaCl(X) \arrow[d] & \times & \mathrm{TF}^*(X) \arrow[r] & \mathrm{TF}^*(X) \\ A^1(V(X)) & \times & A^*(V(X)) \arrow[r] \arrow[u] & A^*(V(X)) \arrow[u] \end{tikzcd}$$

is compatible. We will make this precise in Proposition 1.1.8.

Definition 1.1.7. Assume (Y, ω_Y) is a tropical fan of dimension k in X. The Weil-divisor of a Cartier divisor f on (Y, ω_Y) is a tropical fan (Y_f, ω_f) of dimension $k-1$ in X with the weights being

$$\omega_f(\tau) = \sum_\sigma f_\sigma(u_{\sigma/\tau} \cdot \omega_Y(\sigma)) - f_\tau(\sum_\sigma u_{\sigma/\tau} \cdot \omega_Y(\sigma)),$$

where the sum goes over all k-dimensional cones containing τ as a face and f_σ, f_τ are linear functions equal to f on σ, τ respectively.

The tropical fan (Y_f, ω_f) is well defined. Because of the balancing condition, the choice of f_τ has no effect on $\omega_f(\tau)$. If f vanishes on τ, then the choice of $u_{\sigma/\tau}$ will not have a effect on ω_f. In addition, if Cartier divisor $f \sim g$, then $\omega_f = \omega_g$. We can always find a linear function u on $|X|$ such that $f - u$ vanishes on τ. So the choice of $u_{\sigma/\tau}$ will not affect ω_f. We usually denote the operation as $f \smile Y$ for short.

Proposition 1.1.8. *For $c \in A^*(V(X))$ and the Cartier divisor $D_f \in A^1(V(X))$, we have*

$$Y_{D_f \smile c} = f \smile Y_c.$$

Proof. Let $\tau \in X^{(k)}$ and f_τ be a linear function such that $f = f_\tau$ on τ. Then $D_f = D_{f-f_\tau}$ in $A^1(V(X))$. For $\sigma \in X^{(k+1)}$, pick a representative $\tilde{u}_{\sigma/\tau} \in N$ of $u_{\sigma/\tau}$. The weight of $Y_{D_f \smile c}$ on τ is

$$\begin{aligned}
&D_f \smile c(D_\tau)\\
=&c(D_{f-f_\tau}(D_\tau))\\
=&c(\sum_\sigma (f-f_\tau)(\tilde{u}_{\sigma/\tau})D_\sigma)\\
=&\sum_\sigma f(\tilde{u}_{\sigma/\tau})c(D_\sigma) - \sum_\sigma f_\tau(\tilde{u}_{\sigma/\tau}c(D_\sigma))\\
=&\sum_\sigma f(\tilde{u}_{\sigma/\tau}c(D_\sigma)) - f_\tau(\sum_\sigma \tilde{u}_{\sigma/\tau}c(D_\sigma)),
\end{aligned}$$

where the sum goes over all dimension $k+1$ cones containing τ as a face. Recall $c(D_\sigma)$ is the weight of Y_c on σ. The last equation holds because f_τ is linear. Moreover, the result matches the definition for the weight of $f \smile Y_c$ on τ. □

So the group $CaCl(X)$ acts on $\mathrm{TF}^*(X)$ like the cup product in Chow cohomology $A^*(V(X))$. If f, g are Cartier divisors on X and $Y \in \mathrm{TF}^*(X)$, then $g \smile (f \smile Y) = f \smile (g \smile Y)$. The subgroup of k-dimensional tropical fans is denoted by TF^k.

Example 1.1.9. We will do some intersection theory on X where $V(X) \cong \mathbb{P}^2$. Let N be $\mathbb{Z}^2$ and X be the collection of σ_1, σ_2, σ_3 and their proper faces where $\sigma_1 = \mathbb{R}_{\geq 0}(0,1) + \mathbb{R}_{\geq 0}(1,0), \sigma_2 = \mathbb{R}_{\geq 0}(-1,-1) + \mathbb{R}_{\geq 0}(0,1)$ and $\sigma_3 = \mathbb{R}_{\geq 0}(-1,-1) + \mathbb{R}_{\geq 0}(1,0)$.

For an element (Y, ω) in TF^2, the balancing condition regarding $\tau_1 = \mathbb{R}_{\geq 0}(-1,-1)$ is

$$\omega(\sigma_2) \cdot (0,1) + \omega(\sigma_3) \cdot (1,0) \in \mathbb{R}(-1,-1).$$

So $\omega(\sigma_2) = \omega(\sigma_3)$. The balancing conditions regarding $\tau_2 = \mathbb{R}_{\geq 0}(1,0)$ and $\tau_3 = \mathbb{R}_{\geq 0}(0,1)$ give $\omega(\sigma_1) = \omega(\sigma_3)$ and $\omega(\sigma_1) = \omega(\sigma_2)$. So $\mathrm{TF}^2 \cong \mathbb{Z}$ which is generated by the tropical fan whose weights on σ_1, σ_2 and σ_3 are 1. The generator is the tropical fan associated to the generator $[\mathbb{P}^2] \in A^0(\mathbb{P}^2) \cong \mathbb{Z}$.

For an element (Y, ω) in TF^1, the balancing condition regarding 0 is

$$\omega(\tau_1)(-1,-1) + \omega(\tau_2)(1,0) + \omega(\tau_3)(0,1) = 0.$$

Hence $\omega(\tau_1) = \omega(\tau_2) = \omega(\tau_3)$. So $\mathrm{TF}^1 \cong \mathbb{Z}$ which is generated by the tropical fan whose weights on τ_1, τ_2 and τ_3 are 1. The generator is the tropical fan associated to the generator of $A^1(\mathbb{P}^2) \cong \mathbb{Z}$.

For an element (Y,ω) in TF^0, there is no balancing condition. So $\mathrm{TF}^0 \cong \mathbb{Z}$ which is generated by tropical fan whose weights on 0 are 1. The generator is the tropical fan associated to the generator $[pt] \in A^2(\mathbb{P}^2) \cong \mathbb{Z}$.

Now we perform some intersection theory on X and compare the result with the intersection theory on $V(X)$. Let f on X be the Cartier divisor determined by

$$(1,0) \mapsto 0, (0,1) \mapsto 1, (-1,-1) \mapsto 0.$$

Then f maps to $[D_{\tau_3}]$ under the morphism $CaCl(X) \to A^1(V(X))$. Let $(Y, \omega_Y) \in \mathrm{TF}^2(X)$ be the tropical fan associated to $[\mathbb{P}^2] \in A^0(\mathbb{P}^2)$, i.e., the weights

$$\omega_Y(\sigma_i) = 1, i = 1,2,3.$$

Then the weights $\omega_{f\smile Y}$ on τ_1 is

$$\begin{aligned} &f_{\sigma_2}(\omega_Y(\sigma_2)(0,1)) + f_{\sigma_3}(\omega_Y(\sigma_3)(1,0)) - f_{\tau_1}(\omega_Y(\sigma_2)(0,1) + \omega_Y(\sigma_3)(1,0)) \\ =&1 + 0 - f_{\tau_1}((1,1)) = 1. \end{aligned}$$

Similar calculation shows $\omega_{f\smile Y}(\tau_2) = \omega_{f\smile Y}(\tau_3) = 1$. So $f \smile Y$ is the tropical fan associated to $[D_{\tau_3}] \in A^1(V(X))$. Moreover, the weights $\omega_{f\smile f\smile Y}$ on 0 is

$$\begin{aligned} &f_{\tau_1}(\omega_{f\smile Y}(\tau_1)(-1,-1)) + f_{\tau_2}(\omega_{f\smile Y}(\tau_2)(1,0)) + f_{\tau_3}(\omega_{f\smile Y}(\tau_3)(0,1)) \\ &- f_0(\omega_{f\smile Y}(\tau_1)(-1,-1) + \omega_{f\smile Y}(\tau_2)(1,0) + \omega_{f\smile Y}(\tau_3)(0,1)) \\ =&0 + 0 + 1 - 0 = 1. \end{aligned}$$

So $f \smile f \smile Y$ is the tropical fan associated to $[pt] \in A^2(V(X))$. Both $f \smile Y$ and $f \smile f \smile Y$ matches the intersection theory on $V(X) \cong \mathbb{P}^2$.

1.2 Intersection theory on $\overline{\mathscr{M}}_{0,n}$

In this section we will discuss how calculations in tropical geometry can be applied to the intersection theory on the moduli space $\overline{\mathscr{M}}_{0,n}$.

Usually the moduli space $\overline{\mathscr{M}}_{0,n}$ is not a toric variety. In [18], Gibney and Maclagan show that the moduli space $\overline{\mathscr{M}}_{0,n}$ embeds in a toric variety and apply the projection formula to get the intersection theory of $\overline{\mathscr{M}}_{0,n}$'s boundary divisors.

Definition 1.2.1. [18, Definition 5.1] We define the fan $\mathscr{M}_{0,n}^{\text{trop}}$ as follows. Let

$$\mathscr{I} = \{I \subset [n] : 1 \in I \text{ and } |I|, |I^c| \geq 2\}.$$

Elements in $\mathscr{I}$ can be identified with partitions of the set $[n]$ into two parts with each parts containing at least 2 elements. Then I is the part containing element 1. The simplicial fan $\mathscr{M}_{0,n}^{\text{trop}}$ has 1-dimensional cones labeled by $I \in \mathscr{I}$; a cone σ is a simplex of $\mathscr{M}_{0,n}^{\text{trop}}$ if and only if for all 1-dimensional cones $I, J \in \sigma$ we have $I \subset J$, $J \subset I$ or $I \cup J = [n]$. Let

$$\mathscr{E} = \{ij : 2 \leq i < j \leq n, ij \neq 23\}$$

be an indexing set for a basis of $\mathbb{R}^{\binom{n}{2}-n}$. Define the vector

$$\mathrm{r}_I = (R_{ij,I})_{ij \in \mathscr{E}} \in \mathbb{Z}^{\binom{n}{2}-n}, R_{ij,I} = \begin{cases} 1 & |I \cap \{i,j\}| = 0, |I \cap \{2,3\}| > 0 \\ -1 & |I \cap \{i,j\}| > 0, |I \cap \{2,3\}| = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Form the $(\binom{n}{2} - n) \times |\mathscr{I}|$ matrix R with I-th column r_I.

By the definition 1.2.1 above, the simplicial fan $\mathscr{M}_{0,n}^{\text{trop}}$ embeds into $\mathbb{R}^{\binom{n}{2}-n}$ by mapping the 1-dimensional cone I to $\mathbb{R}_{\geq 0}\mathrm{r}_I$. The image of $\mathscr{M}_{0,n}^{\text{trop}}$ in $\mathbb{R}^{\binom{n}{2}-n}$ is precisely the polyhedral fan Δ in [18].

Let Σ_n be the secondary fan or the saturated Gröbner fan of $\mathscr{M}_{0,n}^{\text{trop}}$ which is the fan Σ^* in [18]. Then Σ_n is complete. Let $\overline{\mathscr{M}}_{0,n}$ be the moduli space of rational stable curves with n marked points. Let the scheme $\mathscr{M}_{0,n} \subset \overline{\mathscr{M}}_{0,n}$ be the moduli space of smooth rational curves with n marked points.

Proposition 1.2.2. *The toric variety $V(\mathscr{M}_{0,n}^{trop})$ is the union of those $T^{\binom{n}{2}-n}$-orbits of $V(\Sigma_n)$ intersecting the closure of $\mathscr{M}_{0,n}$ in this $V(\Sigma_n)$. The closure of $\mathscr{M}_{0,n} \subset T^{\binom{n}{2}-n}$ inside $V(\Sigma_n)$ is equal to $\overline{\mathscr{M}}_{0,n}$.*

Proof. This is a rephrasing of [18, Theorem 5.7]. The embedding $\mathscr{M}_{0,n} \subset T^{\binom{n}{2}-n}$ is in [18, Example 3.1]. □

So the toric variety $V(\mathscr{M}_{0,n}^{\text{trop}})$ contains $\overline{\mathscr{M}}_{0,n}$ as a subvariety. Let D_I be the codimension 1 toric stratum corresponding to the 1-dimensional cone $\mathbb{R}_{\geq 0}\mathrm{r}_I$. Let δ_I be the boundary divisor of $\overline{\mathscr{M}}_{0,n}$ corresponding to curves which can be split into two components by pulling apart at some node, such that the points marked by I are on one component while the points marked by I^c are on the other component.

Proposition 1.2.3. *We have* $\mathrm{Cl}(V(\mathscr{M}_{0,n}^{trop})) \cong \mathrm{Pic}(\overline{\mathscr{M}}_{0,n})$ *with the isomorphism taking the toric strata* $[D_I]$ *to the boundary divisor* δ_I*.*

Proof. The case $n \geq 5$ is precisely [18, Lemma 6.2]. For $n = 4$ we have $\mathrm{Cl}(V(\mathscr{M}_{0,n}^{\mathrm{trop}})) \cong \mathrm{Pic}(\overline{\mathscr{M}}_{0,n}) \cong \mathbb{Z}$ where $[D_I]$ and δ_I are the generators of the groups $\mathrm{Cl}(V(\mathscr{M}_{0,n}^{\mathrm{trop}}))$ and $\mathrm{Pic}(\overline{\mathscr{M}}_{0,n})$ respectively. □

Example 1.2.4. Consider $\overline{\mathscr{M}}_{0,4} \cong \mathbb{P}^1$. The 3 boundary divisors are denoted by δ_{12}, δ_{13} and δ_{14} where δ_{ij} means marked point i and j are on the same curve component. Let

$$\mathscr{I} = \{\{1,2\},\{1,3\},\{1,4\}\}.$$

Let

$$\mathscr{E} = \{ij : 2 \leq i < j \leq 4, ij \neq 23\} = \{24, 34\}$$

be an indexing set for a basis of $\mathbb{R}^2$. Define for $I \in \mathscr{I}$ the vectors in this $\mathbb{R}^2$

$$\begin{aligned} \mathrm{r}_{\{1,2\}} &= (0,1) \\ \mathrm{r}_{\{1,3\}} &= (1,0) \\ \mathrm{r}_{\{1,4\}} &= (-1,-1). \end{aligned}$$

Consider the one-dimensional fan $\Delta = \{\mathbb{R}_{\geq 0}\mathrm{r}_I : I \in \mathscr{I}\}$ in $\mathbb{R}^2$. Then there is an embedding $\overline{\mathscr{M}}_{0,4} \hookrightarrow V(\Delta) \cong \mathbb{P}^2 \backslash ([1:0:0] \cup [0:1:0] \cup [0,0,1])$. Moreover, we have $\mathrm{Cl}(V(\Delta)) \cong \mathrm{Pic}(\overline{\mathscr{M}}_{0,4})$ with the isomorphism taking the toric strata $[D_I]$ to δ_I.

The toric variety $V(\Sigma_n)$ is a proper variety containing $V(\mathscr{M}_{0,n}^{\mathrm{trop}})$ as an open subset because $\mathscr{M}_{0,n}^{\mathrm{trop}}$ is a subfan of Σ_n [18, Proposition 5.5, Proposition 5.6].

Let Δ be the $(n-3)$-dimensional tropical fan corresponding to the cycle $[\overline{\mathscr{M}}_{0,n}] \in A^*(V(\Sigma_n))$. We can label the strata of $\overline{\mathscr{M}}_{0,n}$ by $\sigma \in \mathscr{M}_{0,n}^{\mathrm{trop}}$. To be more precise, we define the stratum

$$\delta_\sigma := \delta_{I_1} \cap \ldots \cap \delta_{I_k}$$

where $r_{I_1}, \ldots, r_{I_k}$ spans δ.

Theorem 1.2.5. *The weight function ω_Δ of Δ on $\sigma \in \Sigma_n^{(n-3)}$ is*

$$\omega_\Delta(\sigma) = \begin{cases} 1, & \sigma \in \mathscr{M}_{0,n}^{trop} \\ 0, & \sigma \notin \mathscr{M}_{0,n}^{trop} \end{cases}.$$

For Cartier divisor $f_i \in \mathrm{CaCl}(\Sigma_n)$ and corresponding boundary divisor $\delta_i \in A^1(\overline{\mathscr{M}}_{0,n})$, the intersection number

$$\prod_{i=1}^{k} \delta_i \cdot \delta_\sigma$$

is equal to the weight of tropical fan

$$f_1 \smile \ldots \smile f_k \smile \Delta$$

on $\sigma \in \Sigma_n$, where $\delta_\sigma \in A_*(\overline{\mathscr{M}}_{0,n})$ is the cycle corresponding to σ.

Proof. The construction of the embedding $j : \overline{\mathscr{M}}_{0,n} \hookrightarrow V(\Sigma_n)$ satisfies that for $\tau \in \Sigma_n$,

$$[\overline{\mathscr{M}}_{0,n}].[D_\tau] \neq 0 \iff \tau \in \mathscr{M}_{0,n}^{\text{trop}}.$$

Moreover, if $\mathrm{r}_{I_1}, \ldots, \mathrm{r}_{I_k}$ spans $\tau \in \Sigma_n^{(k)}$ then $[\overline{\mathscr{M}}_{0,n}].[D_\tau]$ is the cycle corresponding to the subvariety δ_τ of $\overline{\mathscr{M}}_{0,n}$. In particular, if $\tau \in \Sigma_n^{(n-3)}$, then $[\overline{\mathscr{M}}_{0,n}].[D_\tau]$ is either a point or trivial. So the weights of Δ are 1 on $\sigma \in \mathscr{M}_{0,n}^{\text{trop}}$ and 0 otherwise.

Let D_i be the image of f_i under the isomorphism $\mathrm{CaCl}(\Sigma_n) \to A^1(V(\Sigma_n))$. Then δ_i is precisely j^*D_i. For $\sigma \in \Sigma_n$, the cycle class δ_σ is j^*D_σ. For σ of dimension $n-3-k$ applying the projection formula, the intersection number

$$\begin{aligned} & j_*(\prod_{i=1}^{k} \delta_i \cdot \delta_\sigma) \\ =& j_*([\overline{\mathscr{M}}_{0,n}] \cdot j^*D_\sigma \cdot \prod_{i=1}^{k} j^*D_i) \\ =& j_*([\overline{\mathscr{M}}_{0,n}]) \cdot D_\sigma \cdot \prod_{i=1}^{k} D_i \\ =& D_1 \smile \ldots \smile D_k \smile [\overline{\mathscr{M}}_{0,n}](D_\sigma) \end{aligned}$$

is equal to the weight of tropical fan $f_1 \smile \ldots \smile f_k \smile \Delta$ on σ. □

The intersection of geometric cycles on $\overline{\mathscr{M}}_{0,n}$ can be calculated via intersection of toric strata on $V(\Sigma_n)$. So intersecting psi-classes and boundary divisors can be calculated via intersecting Cartier divisors and tropical fans on Σ_n. The intersection of psi-classes on $\mathscr{M}_{0,n}^{\text{trop}}$ is studied in [30].

Now we define the tropical psi-classes and boundary divisors as tropical fans on $\mathscr{M}_{0,n}^{\text{trop}}$. Moreover, we will find the Cartier divisors on Σ_n which give us the psi-classes and boundary divisors so that we can do tropical intersection theory.

Recall that the dual intersection graphs index the strata of $\overline{\mathscr{M}}_{0,n}$. Explicitly, the *dual intersection graph* G of a curve C in $\overline{\mathscr{M}}_{0,n}$ is the graph with

- vertices each labeled by an irreducible component v of C,
- rays marked by $1,\dots,n$ such that the ray marked by i is attached to vertex v if and only if the marked point i lies on the irreducible component v and
- an edge joining vertices v and v' if and only if the irreducible components v and v' have a common node in C.

For a dual intersection graph G and an edge e, we can contract all edges except e to get a graph of two vertices joining by e. Assume the sets of rays attached to the two vertices are I_e and I_e^c respectively where we choose I_e such that $1 \in I_e$. Then $I_e \in \mathscr{I}$ if G is a dual intersection graph of a stable curve in $\overline{\mathscr{M}}_{0,n}$. If $\{I_e | e \text{ is an edge of } G\}$ spans $\sigma \in \mathscr{M}_{0,n}^{\text{trop}}$, then we define the stratum

$$D_G := \delta_\sigma.$$

In addition, we have $D_G = D_\sigma \cap \overline{\mathscr{M}}_{0,n}$ under the embedding $\overline{\mathscr{M}}_{0,n} \hookrightarrow V(\Sigma_n)$. We denote this cone σ_G.

Definition 1.2.6. For $k \in [n]$, the tropical Psi-class $\Psi_k \in \mathrm{TF}^*(\Sigma_n)$ is the $(n-4)$-dimensional tropical fan whose weights are 1 on σ_G and 0 otherwise. Here G runs over all $(n-4)$-edge dual intersection graphs of genus 0, n-marked stable curves whose component containing marked point k contains 4 special points.

A $(n-4)$-edge dual intersection graph G corresponds to a dimension 1 stratum of $\overline{\mathscr{M}}_{0,n}$. The weight on σ_G is the stratum's intersection number with ψ_k which is 1 if G's only 4-valent vertex contains k and 0 if otherwise.

Let $V_k := \{\mathrm{r}_I | I \in \mathscr{I}; k \notin I, |I| = 2 \text{ or } k \in I, |I| = n-2\}$. This is precisely the V_k in [30].

Proposition 1.2.7. *[30, Lemma 2.3] For any $k \in \{1,\dots,n\}$, the linear span of the set V_k equals $\mathbb{R}^{\binom{n}{2}-n}$.*

Proposition 1.2.8. *[30, Lemma 2.4] The sum over all elements $r_I \in V_k$ is* 0 *in* $\mathbb{R}^{\binom{n}{2}-n}$*, i.e.,*

$$\sum_{r_I \in V_k} r_I = 0 \in \mathbb{R}^{\binom{n}{2}-n}.$$

Proposition 1.2.9. *[30, Lemma 2.5] Any element in $\mathbb{R}^{\binom{n}{2}-n}$ has a unique representation by elements in V_k where at least* 1 *coefficient is* 0 *while all the other coefficients are nonnegative, i.e. there is a unique set of λ_I for $v \in \mathbb{R}^{\binom{n}{2}-n}$ such that*

$$v = \sum_{I:\mathrm{r}_I \in V_k} \lambda_I \mathrm{r}_I,$$

where $\lambda_I \geq 0$ and $\lambda_I = 0$ for some I.

The unique representation is called the *positive representation*.

For any $k \in [n]$, define $f_k : \mathbb{R}^{\binom{n}{2}-n} \to \mathbb{R}$ as the linear extension of $V_k \ni \mathrm{r}_I \mapsto 1$. Here the linear extension is via the positive representation, i.e. if the representation is $v = \sum_{I:\mathrm{r}_I \in V_k} \lambda_I \mathrm{r}_I$, then

$$f(v) = \sum_{I:\mathrm{r}_I \in V_k} \lambda_I.$$

Proposition 1.2.10.

$$f_k \smile \Delta = \binom{n-1}{2} \psi_k.$$

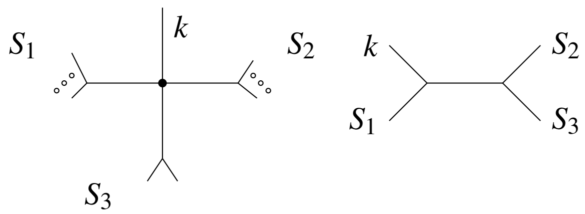
Proof. This is a rephrasing of [30, Proposition 3.5]. The idea will recur later so the sketch of the proof in [30] is shown here.

The tropical fan Δ has weight 0 on $\sigma \notin \mathscr{M}_{0,n}^{\mathrm{trop}}$ so it suffices to consider the $(n-3)$-dimensional cones in $\mathscr{M}_{0,n}^{\mathrm{trop}}$. Let

$$\mathscr{I}_k = \{I | r_I \in V_k\}.$$

Then elements in $\mathscr{I}_k$ can be viewed as dual intersection graphs with two vertices where one vertex has valence 3 while k is attached to the other vertex.

Assume τ is a $(n-4)$-dimensional cone corresponding to the graph G whose only 4-valent vertex contains k. So one of the rays or edges attached to the vertex has to be k. Denote the set of marked points connected to the other 3 edges or rays by S_1, S_2 and S_3. There are exactly three $(n-3)$-dimensional cones σ_1, σ_2 and σ_3 containing τ as a face. Then σ_i is the cone corresponding to adding an extra edge to G such that S_i and k lies on the same vertex while marked points in the other two lies on the other.



graph G graph associated to I_1

Choose the vector r_{I_i} as the representative of $u_{\sigma_i/\tau}$ where

$$I_i = \begin{cases} S_i \cup k, & 1 \in S_i \cup k \\ (S_i \cup k)^c, & 1 \notin S_i \cup k. \end{cases}$$

Notice the representation of r_I by elements in V_k is

$$r_I = \sum_{J \in S_I} r_J$$

where

$$S_I = \begin{cases} \{\{i_1, i_2\}^c | i_1, i_2 \in I^c, i_1 \neq i_2\}, & k \in I \\ \{\{1, i\} | i \in I, i \neq 1\} \cup \{\{i_1, i_2\}^c | i_1, i_2 \in I, i_1 \neq 1, i_2 \neq 1\}, & k \notin I. \end{cases}$$

This is a rephrasing of [30, Lemma 2.7] which promises any element in S_I is a set containing 1. So by definition the weight on τ is

$$\begin{aligned} & f_k(r_{I_1}) + f_k(r_{I_2}) + f_k(r_{I_3}) - f_k(r_{I_1} + r_{I_2} + r_{I_3}) \\ =& \binom{|S_2| + |S_3|}{2} + \binom{|S_1| + |S_3|}{2} + \binom{|S_2| + |S_1|}{2} - (\binom{|S_1|}{2} + \binom{|S_2|}{2} + \binom{|S_3|}{2}) \\ =& \binom{|S_1| + |S_2| + |S_3|}{2} = \binom{n-1}{2}. \end{aligned}$$

Now assume τ is a $(n-4)$-dimensional cone corresponding to the graph G whose only 4-valence vertex does not contain k. Again consider the 4-valence vertex. Denote the set of marked points on each rays or edges attached to the vertex by S_0, S_1, S_2 and S_3 with $k \in S_0$.

So $S_0 \geq 2$. Again there are three $(n-3)$-dimensional cones σ_i containing τ as a face. The for $i = 1,2,3$, choose the representative r_{I_i} of $u_{\sigma_i/\tau}$ as

$$I_i = \begin{cases} S_i \cup S_0, & 1 \in S_i \cup S_0 \\ (S_i \cup S_0)^c, & 1 \notin S_i \cup S_0. \end{cases}$$

Now because $|S_0| \geq 2$, there must be at least one vector in V_k missing in representation of all three r_{I_i}. So the coefficients of representation of $r_{I_1} + r_{I_2} + r_{I_3}$ will be the sum of the coefficients of the representation of r_{I_i}. By definition, f_k will be linear on the space spanned by r_{I_i}. So the weight on τ is 0. $\square$

After this representation of $\binom{n-1}{2}\psi_k$ as tropical fan of f_k, we are able to intersect psi-classes with each other. In particular, when we intersect $n-3$ psi-classes on $\mathscr{M}_{0,n}^{\text{trop}}$, the weight at the origin coincides with the intersection number of the $n-3$ psi-classes on $\overline{\mathscr{M}}_{0,n}$.

Let the boundary divisors δ_I of $\overline{\mathscr{M}}_{0,n}$ be the stratum corresponding to $I \in \mathscr{I}$. If D_I is the codimension 1 stratum in $V(\Sigma_n)$, then $\delta_I = i^* D_I$ where i is the embedding $\overline{\mathscr{M}}_{0,n} \hookrightarrow V(\Sigma_n)$. Let f_I be the piecewise linear function defined by mapping r_I to 1 while mapping all other r_I to 0. The definition of f_I need to extend the value on r_I to the whole space. For more detail, see [7].

Proposition 1.2.11.

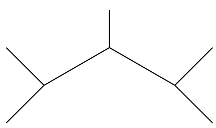
$$f_I \smile \Delta = Y_{\delta_I}$$

where Y_{δ_I} is the tropical fan associated to δ_I.

Proof. By definition, the Cartier divisor f_I maps to D_I under the isomorphism $\mathrm{CaCl}(\Sigma_n) \to A^1(V(\Sigma_n))$. Apply Theorem 1.2.5 to the case δ_I for any $\sigma \in \Sigma_n^{(n-4)}$. The tropical fan $f_I \smile \Delta$ has weight $\delta_I \cdot \delta_\sigma$ which is exactly the weight of Y_{δ_I} on σ $\square$

For examples of intersection numbers of Psi-classes on $\mathscr{M}_{0,5}^{\text{trop}}$ and $\mathscr{M}_{0,6}^{\text{trop}}$, see [32, Example 4.4] and [32, Example 4.5].

Example 1.2.12. We will intersect Psi-classes with boundary divisor on $\overline{\mathscr{M}}_{0,5}$ performed on $\mathscr{M}_{0,5}^{\text{trop}}$. By symmetry, we only show $\psi_1 \smile \delta_{\{1,2\}}$ and $\psi_2 \smile \delta_{\{1,2,3\}}$. The tropical fan Δ associated to the fundamental class $[\overline{\mathscr{M}}_{0,5}]$ assigns weight 1 to each 2-dimensional cone σ_G, where G is a dual intersection graph looking like



.

The set V_1 is

$$\{\mathrm{r}_{\{1,2,3\}}, \mathrm{r}_{\{1,2,4\}}, \mathrm{r}_{\{1,2,5\}}, \mathrm{r}_{\{1,3,4\}}, \mathrm{r}_{\{1,3,5\}}, \mathrm{r}_{\{1,4,5\}}\}.$$

The intersection $\delta_{\{1,2\}} \smile \Delta$ is a 1-dimensional tropical fan. The weight assigned to $\mathrm{r}_{\{1,2,i\}}, i = 3,4,5$ is 1. Indeed, the weight is

$$\begin{aligned}\omega(\mathrm{r}_{\{1,2,i\}}) &= f_{\{1,2\}}(\mathrm{r}_{\{1,2\}}) + f_{\{1,2\}}(\mathrm{r}_{\{1,i\}}) + f_{\{1,2\}}(\mathrm{r}_{\{1,3,4,5\}\setminus i}) - f_{\{1,2\}}(\mathrm{r}_{\{1,2,i\}}) \\ &= 1+0+0-0 = 1.\end{aligned}$$

The weight assigned to $\mathrm{r}_{\{1,2\}}$ is -1. Indeed, the weight is

$$\begin{aligned}\omega(\mathrm{r}_{\{1,2\}}) &= f_{\{1,2\}}(\mathrm{r}_{\{1,2,3\}}) + f_{\{1,2\}}(\mathrm{r}_{\{1,2,4\}}) + f_{\{1,2\}}(\mathrm{r}_{\{1,2,5\}}) - f_{\{1,2\}}(\mathrm{r}_{\{1,2\}}) \\ &= 0+0+0-1 = -1.\end{aligned}$$

The weight assigned to $\mathrm{r}_{\{1,i\}}, i = 3,4,5$ and

$$\mathrm{r}_{\{1,i,j\}}, i \neq j, i,j \in \{3,4,5\}$$

is 0. Indeed, all of the 1-dimensional fans has $f_{\{1,2\}}$ value 0. Moreover, any 2-dimensional fan σ_G contains one of the 1-dimensional fans has $f_{\{1,2\}}$ value 0, where G is a dual intersection graph with 2 bounded edges.

The weight of the cone $\{0\}$ of the intersection $\psi_1 \smile \delta_{\{1,2\}} \smile \Delta$ is 0. Indeed, the positive representation for any $\mathrm{r}_{\{1,2,i\}}$ is itself, so $f_1(\mathrm{r}_{\{1,2,i\}}) = 1, i = 3,4,5$. The positive representation for $\mathrm{r}_{\{1,2\}}$ is $\mathrm{r}_{\{1,2\}} = \mathrm{r}_{\{1,2,3\}} + \mathrm{r}_{\{1,2,4\}} + \mathrm{r}_{\{1,2,5\}}$, so $f_1(\mathrm{r}_{\{1,2\}}) = 3$. The positive representation for $\sum_{i=3}^{5} \mathrm{r}_{\{1,2,i\}} + (-1) \times \mathrm{r}_{\{1,2\}}$ is 0. In total, we have

$$\begin{aligned}\binom{4}{2}\omega(\{0\}) &= \textstyle\sum_{i=3}^{5} f_1(\mathrm{r}_{\{1,2,i\}}) + (-1) \times f_1(\mathrm{r}_{\{1,2\}}) - f_1(\sum_{i=3}^{5} \mathrm{r}_{\{1,2,i\}} + (-1) \times \mathrm{r}_{\{1,2\}}) \\ &= 1+1+1+(-1) \times 3 - 0 = 0.\end{aligned}$$

For the intersection $\delta_{\{1,2,3\}} \smile \Delta$, the weight assigned to $\mathrm{r}_{\{1,i\}}, i = 2,3$ is 1. Indeed, the weight is

$$\begin{aligned}\omega(\mathrm{r}_{\{1,i\}}) &= f_{\{1,2,3\}}(\mathrm{r}_{\{1,2,3\}}) + f_{\{1,2,3\}}(\mathrm{r}_{\{1,i,4\}}) + f_{\{1,2,3\}}(\mathrm{r}_{\{1,i,5\}}) - f_{\{1,2,3\}}(\mathrm{r}_{\{1,i\}}) \\ &= 1+0+0-0 = 1.\end{aligned}$$

The weight assigned to $\mathrm{r}_{\{1,4,5\}}$ is 1. Indeed, the weight is

$$\begin{aligned}\omega(\mathrm{r}_{\{1,4,5\}}) &= f_{\{1,2,3\}}(\mathrm{r}_{\{1,4\}})+f_{\{1,2,3\}}(\mathrm{r}_{\{1,5\}})+f_{\{1,2,3\}}(\mathrm{r}_{\{1,2,3\}})-f_{\{1,2,3\}}(\mathrm{r}_{\{1,4,5\}})\\ &=0+0+1-0=1.\end{aligned}$$

The weight assigned to $\mathrm{r}_{\{1,2,3\}}$ is -1. Indeed, the weight is

$$\begin{aligned}\omega(\mathrm{r}_{\{1,2,3\}}) &= f_{\{1,2,3\}}(\mathrm{r}_{\{1,2\}})+f_{\{1,2,3\}}(\mathrm{r}_{\{1,3\}})+f_{\{1,2,3\}}(\mathrm{r}_{\{1,4,5\}})-f_{\{1,2,3\}}(\mathrm{r}_{\{1,2,3\}})\\ &=0+0+0-1=-1.\end{aligned}$$

The weight assigned to $\mathrm{r}_{\{1,4\}},\mathrm{r}_{\{1,5\}}$ and

$$\mathrm{r}_{\{1,i,j\}}, i\in\{2,3\}, j\in\{4,5\}$$

is 0 for similar reason. Indeed, all of the 1-dimensional fans has $f_{\{1,2,3\}}$ value 0. Moreover, any 2-dimensional fan σ_G contains one of the 1-dimensional fans has $f_{\{1,2,3\}}$ value 0, where G is a dual intersection graph with 2 bounded edges.

The weight of the cone $\{0\}$ of the intersection $\psi_1 \smile \delta_{\{1,2,3\}} \smile \Delta$ is 1. Indeed, the positive representation for any $\mathrm{r}_{\{1,i,j\}}$ is itself for $i\neq j$, so $f_1(\mathrm{r}_{\{1,4,5\}})=f_1(\mathrm{r}_{\{1,2,3\}})=1$. The positive representation for $\mathrm{r}_{\{1,i\}}, i=2,3$ is $\mathrm{r}_{\{1,i\}}=\mathrm{r}_{\{1,2,3\}}+\mathrm{r}_{\{1,i,4\}}+\mathrm{r}_{\{1,i,5\}}$, so $f_1(\mathrm{r}_{\{1,i\}})=3$. The positive representation for $\mathrm{r}_{\{1,2\}}+\mathrm{r}_{\{1,3\}}+\mathrm{r}_{\{1,4,5\}}+(-1)\times\mathrm{r}_{\{1,2,3\}}$ is 0. In total, we have

$$\begin{aligned}\binom{4}{2}\omega(\{0\}) &= \textstyle\sum_{i=2}^{3} f_1(\mathrm{r}_{\{1,i\}})+f_1(\mathrm{r}_{\{1,4,5\}})+(-1)\times f_1(\mathrm{r}_{\{1,2,3\}})\\ &\quad -f_1(\textstyle\sum_{i=2}^{3}\mathrm{r}_{\{1,i\}}+\mathrm{r}_{\{1,4,5\}}+(-1)\times\mathrm{r}_{\{1,2,3\}})\\ &= 3+3+1+(-1)-0=6.\end{aligned}$$

All the calculation matches the classical calculation on $\overline{\mathscr{M}}_{0,5}$ where $\psi_1 \smile \delta_{\{1,2\}}=0$ and $\psi_1 \smile \delta_{\{1,2,3\}}=1$.

Chapter 2

The moduli space of twisted curves $\overline{\mathscr{M}}_\eta$

In this chapter we will recall the definition of the moduli stack $\overline{\mathscr{M}}_\eta$ of étale covers. We then focus on the intersection theory on $\overline{\mathscr{M}}_\eta$. In particular, we are interested in the relation between the intersection theory on $\overline{\mathscr{M}}_\eta$ and the intersection theory on $\overline{\mathscr{M}}_{0,n}$.

2.1 The moduli stack $\overline{\mathscr{M}}_\eta$

In this section we recall the definition of $\overline{\mathscr{M}}_\eta$ and some related results.

Let I be a finite set with n elements and let $m : I \to \mathbb{Z}_{>0}$ be a function.

Definition 2.1.1. A genus g *(balanced) twisted nodal n-marked curve* over a scheme U is a diagram

$$\begin{tikzcd}
 & & \mathscr{C} \arrow[r] \arrow[dr] & C \arrow[d] \\
U \times I \arrow[r] & U \times \coprod_{i\in I} B\mu_{m(i)} \arrow[ur, hook] \arrow[rr] & & U
\end{tikzcd}$$

where

- scheme U is of finite type.
- The stack $\mathscr{C}$ is a proper separated flat DM stack over U, étale locally a nodal curve over U.
- The map $\mathscr{C} \to C$ exhibits C as the coarse moduli space of $\mathscr{C}$, and C is connected of genus g.
- The morphism $U \times \coprod_{i\in I} B\mu_{m(i)} \hookrightarrow \mathscr{C}$ is an embedding of a disjoint union of trivial $\mu_{m(i)}$-gerbes into $\mathscr{C}$, and $U \times I \to U \times \coprod_{i\in I} B\mu_{m(i)}$ are sections of these gerbes.

- The morphism $\mathscr{C} \to C$ is an isomorphism away from the nodes and marked points of C.
- Étale locally near a node of $\mathscr{C}$, the morphism $\mathscr{C} \to U$ looks like

$$(\operatorname{Spec} A[u,v]/(uv-t))/\mu_r \to \operatorname{Spec} A$$

where $r \in \mathbb{N}, t \in A$ while the group of r-th roots of unity μ_r acts on $A[u,v]/(uv-t)$ by $u \to lu, v \to l^{-1}v$ with $l \in \mu_r$.

This definition comes from [12] and is originally due to [4] with more details in [5]. Intuitively, a twisted nodal curve is an ordinary curve with the nodes and marked points replaced by

$$(\operatorname{Spec} A[u,v]/(uv-t))/\mu_r \to \operatorname{Spec} A$$

and

$$(\operatorname{Spec} A[u])/\mu_{m(i)} \to \operatorname{Spec} A.$$

The *multiplicity* is the value $m(i)$ at the marked point i and r at the node locally being $(\operatorname{Spec} A[u,v]/(uv-t))/\mu_r$.

A twisted nodal curve $\mathscr{C}$ is *stable* if its coarse moduli space C is, i.e., on each irreducible curve component D of C,

$$2g_D - 2 + \#\text{Nodes} + \#\text{Marked points} > 0.$$

Definition 2.1.2. The moduli stack $\overline{\mathscr{M}}_{g,I,m}$ is the category of genus g stable twisted nodal curves with n points marked by I and orbifold structure given by m.

This is the same as the moduli stack $\mathfrak{M}_{g,I,m,a}$ in [12, Chapter 2] if we take $A = 0, a = 0$. Let $\overline{\mathscr{M}}_{g,I}$ be the Deligne-Mumford moduli space of stable genus g, n-marked curves where $n = \#I$. For a standard introduction about $\overline{\mathscr{M}}_{g,I}$, see [15]. There is a map

$$\overline{\mathscr{M}}_{g,I,m} \to \overline{\mathscr{M}}_{g,I}$$

which associates to a twisted curve its coarse moduli space.

We need some extra notation in order to label the stack of étale covers of twisted nodal curves and some substacks of $\overline{\mathscr{M}}_{g,I,m}$. We will introduce twisted graphs, which are ordinary graphs with extra functions recording the orbifold structure. When used as labels, the twisted graphs are used to specify the dual intersection graphs of twisted nodal marked curves. The twisted graphs will also serve as the underlying graphs of twisted tropical curves, which we will discuss later.

Definition 2.1.3. A *graph* G consists of the data:

- two finite sets vertices $V(G)$ and flags $F(G)$;
- a root map $r_G : V(G) \sqcup F(G) \to V(G)$ which is identity on $V(G)$; and
- an involution $\iota_G : V(G) \sqcup F(G) \to V(G) \sqcup F(G)$ which is identity on $V(G)$.

This definition is originally due to [40] and expanded in [9]. The involution ι_G partitions $F(G)$ into sets $\{f, \iota_G(f)\}$ of size 1 or 2. The sets of size 1 form the unbounded edge (or leg) set $L(G)$ while the sets of size 2 form the edge set $E(G)$. A *loop* is a finite edge $\{f, \iota_G(f)\}$ such that $r_G(f) = r_G(\iota_G(f))$.

A *morphism of graphs* $g : G_1 \to G_2$ is a map of $V \sqcup F$ preserving the graph structure. Explicitly, the morphism g commutes with r, ι and maps legs to legs. Hence a morphism g must map vertices to vertices, legs to legs and edges to edges or vertices. If one flag of an edge maps to a vertex, then the other flag of the edge must map to the same vertex because the morphism commutes with ι. If a flag maps to a vertex, then the flag's root maps to the same vertex because the morphism commutes with r.

A *vertex weighting* on a graph G is a function $h : V(G) \to \mathbb{N}$. The genus of the weighted graph (G,h) is G's first Betti number plus the weight at each vertex, i.e.,

$$g(G,h) = b_1(G) + \sum_{v \in V(G)} h(v).$$

A *marking* on a graph G by a finite set I is a bijection

$$I \xrightarrow{\cong} L(G).$$

The type (g,I) of a weighted marked graph consists of the genus g and the marking I. Two graphs are of the same type if they are of the same genus and are marked by the same finite set. For $n \in \mathbb{N}_{>0}$, define

$$[n] := \{1, ..., n\}.$$

The type $(g,[n])$ is usually denoted by (g,n) when there is no ambiguity.

A *morphism* of weighted marked graphs is a morphism of the underlying graphs.

A *contraction* $f : (G_1, h_1, I) \twoheadrightarrow (G_2, h_2, I)$ is a morphism of weighted marked graphs of the same type such that

- the underlying map $f_G : G_1 \to G_2$ is a graph morphism and is surjective;
- the underlying map f_G commutes with the marking;
- the weight $h_2(v_2)$ on vertex v_2 is the genus of $f_G^{-1}(v_2)$ under the weighting h_1.

A contraction contracts some edges and their roots to one vertex as the name indicates. The set of contracted edges are denoted by $E(f)$.

The *dual intersection graph* of a genus g nodal curve C marked by set I is a weighted marked graph of type (g,I) whose vertex set V is the set of irreducible components of C; leg set L is the marking set I; edge set E is the set of nodes. More explicitly, the flag set is the union of L and two copies of E. If the node e is contained in two components, then the roots of the two copies of e are the two corresponding components in V. If the node e is contained in only one component, then the roots of the two copies of e are both the corresponding component in V. The root of the leg l is the component in V containing the marked points l. The involution maps l to l and maps one copy of e to the other. The weighting on V is the genus of the component. The marking is the identity $I \xrightarrow{=} L$.

In the perspective of dual intersection graph, a contraction is the process of smoothing the nodes corresponding to the contracted edges. A nodal curve C is connected if and only if the dual intersection graph of C is connected. The type of a nodal curve C is the same as the type of the dual intersection graph of C.

Definition 2.1.4. A twisted graph is a connected weighted, marked graph (G,h,I) with an extra function $m : I \sqcup E(G) \to \mathbb{N}$.

The *coarse moduli graph* $r(\Gamma)$ of a twisted graph Γ is simply the underlying weighted marked graph without the extra function. The genus of a twisted graph Γ is the genus of $r(\Gamma)$.

A *morphism of twisted graphs* $f : \Gamma_1 = (G_1,h_1,I_1,m_1) \to \Gamma_2 = (G_2,h_2,I_2,m_2)$ is a morphism of underlying weighted marked graphs $r(f) : r(\Gamma_1) \to r(\Gamma_2)$. The type of a twisted graph $\Gamma = (G,h,I,m)$ is the triple $(g,I,m|_I : I \to \mathbb{N}_{>0})$, i.e., the type (g,I) of $r(\Gamma)$ with the restriction of m on the set I. A *contraction of twisted graphs* is a morphism of twisted graphs of the same type which is a contraction on the coarse moduli graphs. The set of contracted edges of the contraction f is again denoted by $E(f)$.

The *dual intersection graph* of a twisted nodal curve $\mathscr{C}$ marked by I is the dual intersection graph $\Gamma = (G,h,I)$ of the coarse moduli space C with the extra function $m : I \sqcup E(G) \to \mathbb{N}_{>0}$ such that $m|_I$ is the same as the function $I \to \mathbb{N}_{>0}$ defining $\mathscr{C}$ and $m(e) = r$ if étale locally near the node corresponding to e, the twisted curve $\mathscr{C}$ looks like

$$(\operatorname{Spec} A[u,v]/(uv-t))/\mu_r.$$

The extra function m indicates the orbifold structures at nodes and marked points. Similar to the case of nodal curves, the type of the dual intersection graph of a twisted nodal curve $\mathscr{C}$ is the same as the type of the twisted nodal curve $\mathscr{C}$.

The valence $\mathrm{Val}(v)$ of a vertex v is the number of flags rooting at v. A weighted marked graph (G,h,I) is stable if at each vertex $v \in V(G)$,

$$2h(v) - 2 + \mathrm{Val}(v) > 0.$$

A twisted graph Γ is stable if and only if the coarse moduli space $r(\Gamma)$ is stable. By definition, a marked nodal curve or a twisted marked nodal curve is stable if and only if its dual intersection graph is stable.

Let V be a DM stack and $\mathrm{B}S_n$ be the classifying stack of the symmetric group S_n.

Proposition 2.1.5. *The 2-groupoid* $\mathrm{HomRep}(V, \mathrm{B}S_n)$ *of representable maps* $V \to \mathrm{B}S_n$ *is equivalent to the 2-groupoid of proper, separated surjective, and étale of degree n morphism* $V' \to V$ *with the inertia groups of* V *acting faithfully on the fibres of* $V' \to V$.

Proof. This is a special case of [12, Corollary 2.2.2]. We recall the correspondence to help understanding.

Giving a map $V \to \mathrm{B}S_n$ is the same as giving an S_n-bundle $P \to V$. Define

$$V' := P \times^{S_n} \{1, ..., n\}.$$

Then $V' \to V$ is the étale morphism.

Given a proper, separated surjective, and étale of degree n morphism $V' \to V$, one recovers P from V' as the sheaf on the small étale site of V,

$$P := \mathrm{Iso}_{et}(V', V \times \{1, ..., n\}).$$

Then P is an étale locally trivial principal S_n-bundle over V, which gives a morphism $V \to \mathrm{B}S_n$. □

Example 2.1.6. Consider $V' \to V$ where V is the classifying stack $\mathrm{B}\mathbb{Z}_2$. Assume first V' is $V \times \{1,2\}$ with the morphism being the first projection. The associated map $V \to \mathrm{B}S_2$ is not representable. Indeed, we have the Cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & V \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{B}S_2. \end{tikzcd}$$

When $n = 2$, we have $P \cong V'$ which is not an algebraic space. Let us look at the fibered product $\mathrm{pt} \times_V V' = \mathrm{pt} \times \{1,2\}$. The inertia group $\mathbb{Z}_2$ acts trivially on the underlying point. Because $V' \to V$ is trivial, the inertia group action on $\mathrm{pt} \times \{1,2\}$ is trivial hence not faithful on the fibers of $V' \to V$.

Now assume $V' \to V$ is from a point to $\mathrm{B}\mathbb{Z}_2$. If we identify $\mathbb{Z}_2$ with S_2, then the associated map $V \to \mathrm{B}S_2$ is an isomorphism which is representable. Indeed, the same Cartesian diagram shows the map $P \to \mathrm{pt}$ is an isomorphism because $P \cong V'$ is a point now. The fibered product $\mathrm{pt} \times_V V'$ is again two points. But the non-identity element of the inertia group $\mathbb{Z}_2$ sends one point to another. Hence the inertia group acts faithfully on the fibers of $V' \to V$.

Proposition 2.1.7. *Given m_i such that $m_i | m_t, i = 1,2,...,k$, there is a natural morphism*

$$\coprod_i \mathrm{B}\mathbb{Z}_{m_i} \to \mathrm{B}\mathbb{Z}_{m_t}.$$

Then the associated morphism

$$\mathrm{B}\mathbb{Z}_{m_t} \to \mathrm{B}S_n, n := \sum_i \frac{m_t}{m_i}$$

is representable if and only if the least common multiple $lcm(\frac{m_t}{m_1},...,\frac{m_t}{m_k})$ is equal to m_t.

Proof. Let d_i be $\frac{m_t}{m_i}$. Consider the fibered product $\{1,...,d_i\} = \mathrm{pt} \times_{\mathrm{B}\mathbb{Z}_{m_t}} \mathrm{B}\mathbb{Z}_{m_i}$. The inertia group $\mathbb{Z}_{m_t}$ acts on the set $\{1,...,d_i\}$ as $1 \to (12...d_i) \in S_{d_i}$. So the inertia group acts faithfully on the fibers of $\coprod_i \mathrm{B}\mathbb{Z}_{m_i} \to \mathrm{B}\mathbb{Z}_{m_t}$ if and only if $lcm(d_1,...,d_k)$ is equal to m_t, i.e., the least common multiple $lcm(\frac{m_t}{m_1},...,\frac{m_t}{m_k})$ is equal to m_t. □

Definition 2.1.8. An *étale cover* over U is a degree n étale surjective morphism $f : \mathscr{C}' \to \mathscr{C}$ between two twisted nodal curves marked by I', I respectively over U such that

- the diagram

$$\begin{tikzcd} I' \times U \arrow[r] \arrow[d] & \mathscr{C}' \arrow[d] \\ I \times U \arrow[r] & \mathscr{C} \end{tikzcd}$$

is Cartesian over U;

- the map $\mathscr{C} \to \mathrm{B}S_n$ associated to the étale map $\mathscr{C}' \to \mathscr{C}$ is representable.

The condition of Cartesian diagram implies that the marked points of $\mathscr{C}'$ are precisely the points lying over marked marked points of $\mathscr{C}$. That f is étale implies nodes of $\mathscr{C}'$ are the points lying over nodes of $\mathscr{C}$.

Definition 2.1.9. A graph cover $(G,h,I,m) \to (G',h',I',m')$ is a morphism of twisted graphs $\pi : G \to G'$ such that

- it maps vertices to vertices and flags to flags;
- (Degree) define the degree $d(v,e')$ of flag $e' \in F(G')$ with respect to $v \in \pi^{-1}(r_{G'}(e'))$ to be
$$\sum_{e \in F(v) \cap \pi^{-1}(e')} \frac{m'(e')}{m(e)}.$$
Then $d(v,e'_1) = d(v,e'_2)$ if $r_{G'}(e'_1) = r_{G'}(e'_2)$ or $e'_1 = \iota_{G'}(e'_2)$. Define $d(v,v') := d(v,e')$, where $r(e') = v'$;
- (Hurwitz Formula) for each v and $v' = \pi(v)$, the covering satisfies
$$2h(v) - 2 = d(v,v')(2h'(v') - 2) + \sum_{e \in F(v)} (\frac{m'(e')}{m(e)} - 1);$$
- (Representability) the multiplicity of flag $m'(e')$ on the target curve is the least common multiple of $\frac{m'(e')}{m(e)}$, where $e \in \pi^{-1}(e')$.

The *degree* of a graph cover is the sum $\sum_{v \in \pi^{-1}(v')} d(v,v')$ for any fixed $v' \in V(G')$.

The conditions for graph cover come from étale cover. Indeed, to an étale cover $f : \mathscr{C}' \to \mathscr{C}$, we can associate a natural morphism $\Gamma' \to \Gamma$ of the dual intersection graphs of $\mathscr{C}'$ and $\mathscr{C}$. The morphism $\Gamma' \to \Gamma$ is called the *dual intersection graph* of $\mathscr{C}' \to \mathscr{C}$.

Proposition 2.1.10. *The dual intersection graph $\Gamma' \to \Gamma$ of étale cover $f : \mathscr{C}' \to \mathscr{C}$ is a graph cover.*

Proof. The preimage of each geometric point of $\mathscr{C}$ has the same cardinality when counted with multiplicity because f is étale and proper. The multiplicity at point $e' \in f^{-1}(e)$ is exactly $\frac{m(e)}{m'(e')}$. So the degree condition holds.

The morphism $C' \to C$ of coarse moduli spaces of $\mathscr{C}' \to \mathscr{C}$ satisfies the Hurwitz Formula. So the Hurwitz Formula condition holds.

The representability requirement originates from the fact that étale covers satisfy the representable condition in the Definition 2.1.8. Indeed, the representable condition states the morphism $\mathscr{C} \to BS_n$ associated to $\mathscr{C}' \to \mathscr{C}$ is representable, which is equivalent to the statement involving least common multiple by Proposition 2.1.7. □

Given a graph cover Γ, usually we denote the source twisted graph by Γ_s and the target twisted graph by Γ_t.

A *contraction* $f : \Gamma' \twoheadrightarrow \Gamma$ of graph covers is a commutative diagram of twisted graph morphisms

$$\begin{tikzcd} \Gamma'_s \arrow[r, two heads, "f_s"] \arrow[d, "\pi'"] & \Gamma_s \arrow[d, "\pi"] \\ \Gamma'_t \arrow[r, two heads, "f_t"] & \Gamma_t \end{tikzcd}$$

where both f_s and f_t are contractions and $\pi'^{-1}(E(f_s)) = E(f_t)$. The convention is $E(f) := E(f_t)$.

Definition 2.1.11. Given a weighted marked graph (respectively twisted graph, graph cover) η, the moduli stack $\overline{\mathscr{M}}_\eta$ is the category of nodal curves (respectively twisted nodal curves, étale covers) C such that the dual intersection graph Γ of each geometric fiber of C has a contraction $\Gamma \twoheadrightarrow \eta$.

Remark 2.1.12. The moduli stacks defined here appear in [12] as well. Given a graph cover η where η_t has 1 vertex and no edges, the moduli stack $\overline{\mathscr{M}}_\eta$ defined here is precisely the moduli stack $\mathfrak{M}_\eta$ in [12]. For general η, the moduli stack $\overline{\mathscr{M}}_\eta$ is the closed substack of $\overline{\mathscr{M}}_{\eta_0}$ where η_0 is obtained via contracting all the edges of η. Let $f : \eta \to \eta_0$ be the contraction. Then the moduli stack $\overline{\mathscr{M}}_\eta$ here is denoted as $\mathfrak{M}_f$ in [12].

The moduli stack $\overline{\mathscr{M}}_G$ can represent the moduli stack $\overline{\mathscr{M}}_{g,n}$ or $\overline{\mathscr{M}}_{g,I,m}$ if G is chosen carefully. Indeed, the moduli stack $\overline{\mathscr{M}}_G$ is precisely the moduli stack $\overline{\mathscr{M}}_{g,n}$ of genus g, n-marked stable curves if G is the weighted marked graph with only 1 vertex v weighting g and precisely n flags rooting at v marked by $[n]$. Similarly, the moduli stack $\overline{\mathscr{M}}_G$ is the moduli stack $\overline{\mathscr{M}}_{g,I,m}$ if G is the twisted graph with the flags marked by I attaching to the only vertex weighting g; the extra function on G matches m.

Theorem 2.1.13. *Let H be a weighted marked graph, twisted graph or graph cover. Then the moduli stack $\overline{\mathscr{M}}_H$ is algebraic.*

Proof. Let $H \twoheadrightarrow \eta$ be the contraction where η has no edges. If H is a weighted marked graph, then $\overline{\mathscr{M}}_\eta$ is $\overline{\mathscr{M}}_{g,n}$ for some g and n, clearly algebraic. If H is a twisted graph, then $\overline{\mathscr{M}}_\eta$ is algebraic by [4, Theorem 2.1.7]. If H is a graph cover, then $\overline{\mathscr{M}}_\eta$ is algebraic by [12, Proposition 2.3.1].

We will define the functor $i : \overline{\mathscr{M}}_H \to \overline{\mathscr{M}}_\eta$ making $\overline{\mathscr{M}}_H$ a subcategory of $\overline{\mathscr{M}}_\eta$. Any family in $\overline{\mathscr{M}}_H$ is naturally a family in $\overline{\mathscr{M}}_\eta$ because the composition of contractions is a contraction. We define i as the natural map $\overline{\mathscr{M}}_H \to \overline{\mathscr{M}}_\eta$. Moreover, the moduli stack $\overline{\mathscr{M}}_H$ is a closed substack of $\overline{\mathscr{M}}_\eta$ under i. Indeed, given a morphism from a scheme $U \to \overline{\mathscr{M}}_\eta$, the fiber product $U \times_{\overline{\mathscr{M}}_\eta} \overline{\mathscr{M}}_H$ is exactly the closed subscheme of U where the dual intersection graph of the geometric fiber has a contraction to H. So $\overline{\mathscr{M}}_H$ is algebraic. □

Proposition 2.1.14. *The moduli stack $\overline{\mathscr{M}}_G$ is a closed substack of $\overline{\mathscr{M}}_H$ if there is a contraction $G \to H$.*

Proof. The moduli stack $\overline{\mathscr{M}}_G$ is a strictly full subcategory of $\overline{\mathscr{M}}_H$. Given a morphism from a scheme $U \to \overline{\mathscr{M}}_H$, the fiber product $U \times_{\overline{\mathscr{M}}_H} \overline{\mathscr{M}}_G$ is the scheme where the dual intersection graph of the geometric fiber has a contraction to G which is a closed subscheme of U. □

There are several natural morphisms between the moduli stacks $\overline{\mathscr{M}}_\eta$.

Given a graph cover Γ, the morphism $s : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{\Gamma_s}$ maps the étale cover $\mathscr{C}' \to \mathscr{C}$ to the source twisted nodal curve $\mathscr{C}'$. The morphism $t : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{\Gamma_s}$ maps the étale cover $\mathscr{C}' \to \mathscr{C}$ to the target twisted nodal curve $\mathscr{C}$.

Given a twisted graph G whose coarse moduli graph is $r(G)$, the morphism $r : \overline{\mathscr{M}}_G \to \overline{\mathscr{M}}_{r(G)}$ map the twisted nodal curve $\mathscr{C}$ to its coarse moduli curve C.

Proposition 2.1.15. *Given a graph cover H with no edges, the target map $t : \overline{\mathscr{M}}_H \to \overline{\mathscr{M}}_{H_t}$ is étale .*

Proof. This is a rephrasing of [12, Proposition 2.3.1]. □

2.2 Hurwitz numbers

In this section we review Hurwitz numbers and the degeneration formula. The Hurwitz numbers play an important role in pulling back cycles from $\overline{\mathscr{M}}_{G_t}$ to $\overline{\mathscr{M}}_G$.

In [26], Hurwitz initiated the study on coverings of the form $C \to \mathbb{P}^1$ where C is another curve. Hurwitz numbers count the coverings with specified multiplicity conditions. Hurwitz numbers can be calculated as intersections of natural classes on $\overline{\mathscr{M}}_{g,n}$ [13]. Hurwitz numbers are crucial in the calculation of the degree $\overline{\mathscr{M}}_G \to \overline{\mathscr{M}}_{G_t}$.

Assume $\pi : D \to C$ is a cover of connected smooth curves. Given a partition $\mu = (\mu_1, ..., \mu_s)$ and $p \in C$ with $\pi^{-1}(p)$ marked by $q_1, ..., q_t$, the ramification profile over p is μ if and only if $s = t$ and the ramification is μ_i at each q_i.

Definition 2.2.1. Fix r points $p_1, ..., p_r$ on a smooth genus 0 curve C and r partitions $\mu^1, ..., \mu^r$ of an integer $d > 0$. The *local Hurwitz number*

$$h(\mu^1, ..., \mu^r)$$

is the weighted number of degree d covers of smooth connected curves $\pi : D \to C$ with ramification profile μ^i over p_i and no ramifications elsewhere. Each cover is weighted by $1/|\mathrm{Aut}(\pi)|$.

Notice that all $\pi^{-1}(p_i)$ on D are marked as well. An element in $|\mathrm{Aut}(\pi)|$ has to map the marked points to themselves. This definition is the same as the definition in [10, 3.2.4], i.e. the local Hurwitz number $h_{g\to 0,d}(\vec{\mu})$. The only difference between the classical Hurwitz numbers and the local Hurwitz numbers is whether D is marked. The classical Hurwitz number is the local Hurwitz number divided by the number of ways to mark. For example, the partition $(2,2)$ has 2 choices of marking and the partition $(3,2,2,2)$ has 6 choices given by different ways of marking the 3 points of ramification 2.

There is close connection between classical Hurwitz numbers and symmetric groups through monodromy representations. For more detail, see [11]. The classical Hurwitz numbers can be calculated through combinatorics of symmetric groups. Moreover, the dual intersection graph of $D \to C$ contains all the information for the local Hurwitz numbers. We have the following equivalent definition.

Let $G = (\pi : G_s \to G_t)$ be a graph cover where G_t is of genus 0 with no edges and n legs. This implies G_t and G_s both have only 1 vertex v_s and v_t respectively. For any flag f of v_t, the integer sequence

$$(\frac{m(f)}{m(f')}), \forall f' \in \pi^{-1}(f)$$

defines a partition of the degree $d(v_s)$ of $v_s \to v_t$, and hence a conjugacy class $C_f \subset S_{d(v_s)}$. Let $\chi(v_s)$ be the set of tuples

$$\{(\sigma_{f_1},...,\sigma_{f_n})|\sigma_{f_i} \in C_{f_i}, \prod_{i=1}^{n} \sigma_{f_i} = \mathrm{id}, \text{ the subgroup generated by the } \sigma_{f_i} \text{ is transitive}\}.$$

Definition 2.2.2. Let Q be a set of some graphs and some functions on legs. Define the automorphism groups $\mathrm{Aut}(G|Q)$ as the group of automorphisms of G which fix the elements in Q.

The G can be a graph, a twisted graph or a graph cover. For example, given a graph cover G and $Q = \{G_t, m_s\}$, the group $\mathrm{Aut}(G|Q) = \mathrm{Aut}(G|G_t, m_s)$ consists of the automorphisms of G which is identity on G_t and preserves the twisted degree m_s on G_s. The order $\#\mathrm{Aut}(G|G_t, m_s)$ is precisely the number of different possible source curve markings given a ramified covering of the marked sphere.

Definition 2.2.3. The local Hurwitz number $H(v_s)$ is

$$\frac{\#\chi(v_s) \cdot \#\mathrm{Aut}(G|G_t, m_s)}{d(v_s)!}.$$

The part $\frac{\#\chi(v_s)}{d(v_s)!}$ is equal to the classical Hurwitz number.

Now assume G_t is of genus 0. In particular, the roots of any two legs forming an edge are different, i.e. there is no loop edge in G_t. Define the local Hurwitz number

$$H(v) := \prod_{v_s \in \pi^{-1}(v)} H(v_s),$$

where v is a vertex of G_t. The local Hurwitz number $H(v_t)$ is calculated with all legs of v_s and v_t marked, i.e., the number $H(v_t)$ counts different marked coverings of type $\coprod v_s \to v_t$.

In consistency with Definition 2.2.2, the automorphism group $\mathrm{Aut}(G|L(G_s), L(G_t), m_s, m_t)$ is the group of automorphisms of the graph cover $G_s \to G_t$ which are the identity on the legs $L(G_t)$ and $L(G_s)$ and commute with the twisted degree m_s and m_t. When $g(G_t)$ is equal to 0, the group $\mathrm{Aut}(G|L(G_s), L(G_t), m_s, m_t)$ is the same as the group $\mathrm{Aut}(G|G_t, L(G_s), m_s)$ which contains automorphisms fixing G_t, $L(G_s)$ and m_s because G_t is fixed if and only if $L(G_t)$ and m_t are. There are exactly $\#\mathrm{Aut}(G|G_t, L(G_s), m_s)$ ways to label the edges of G_s. Let

$$d_G := \frac{\prod_{e \in E(G_t)} m(e)}{\#\mathrm{Aut}(G|L(G_s), L(G_t), m_s, m_t)} \prod_{v \in V(G_t)} H(v).$$

We will see d_G is the degree for the étale morphism $t : \overline{\mathscr{M}}_G \to \overline{\mathscr{M}}_{G_t}$.

Proposition 2.2.4. *The pushforward of* $[\overline{\mathscr{M}}_G]$ *under the map* $r \circ t$ *is*

$$r_* t_* [\overline{\mathscr{M}}_G] = \frac{d_G}{\prod_{l \in L(G_t)} m(l) \cdot \prod_{e \in E(G_t)} m(e)} [\overline{\mathscr{M}}_{r(G_t)}].$$

Proof. Let the graph cover $G(v)$ be the graph with vertices $\coprod_{v_s \in \pi^{-1}(v)} v_s \to v$ and flags rooting at v_s and v given a vertex $v \in V(G_t)$.

There is a commutative diagram of morphisms

$$\begin{tikzcd}
\prod_{v \in V(G_t)} \overline{\mathscr{M}}_{G(v)} \arrow[r, "t'"] \arrow[d, "f"'] & \prod_{v \in V(G_t)} \overline{\mathscr{M}}_{G(v)_t} \arrow[r, "r'"] \arrow[d, "g"'] & \prod_{v \in V(G_t)} \overline{\mathscr{M}}_{r(G(v)_t)} \arrow[d, "h"'] \\
\overline{\mathscr{M}}_G \arrow[r, "t"'] & \overline{\mathscr{M}}_{G_t} \arrow[r, "r"'] & \overline{\mathscr{M}}_{r(G_t)}.
\end{tikzcd}$$

By [12, Chapter 4], the vertical morphisms are étale of degree

$$||f|| = \frac{\#\mathrm{Aut}(G|L(G_s), L(G_t), m_s, m_t)}{\prod_{e \in E(G_t)} m^2(e)},$$

$$||g|| = \frac{\#\mathrm{Aut}(G_t|L(G_t), m_t)}{\prod_{e \in E(G_t)} m(e)}$$

and

$$||h|| = 1.$$

The morphism t' is étale of degree

$$\prod_{v \in V(G_t)} H(v).$$

Recall that $\#\mathrm{Aut}(G_t|L(G_t), m_t) = 1$ when G_t is of genus 0. So t is of degree

$$\frac{\prod_{e \in E(G_t)} m(e)}{\#\mathrm{Aut}(G|L(G_s), L(G_t), m_s, m_t)} \prod_{v \in V(G_t)} H(v) = d_G.$$

The degree of r' is

$$\prod_{v \in V(G_t)} \prod_{l \in L(G(v)_t)} \frac{1}{m(l)} = \prod_{l \in L(G_t)} \frac{1}{m(l)} \cdot \prod_{e \in E(G_t)} \frac{1}{m^2(e)}$$

by the automorphism calculation [12, Section 2.6]. So

$$r_* t_* [\overline{\mathscr{M}}_G] = \frac{d_G}{\prod_{l\in L(G_t)} m(l) \cdot \prod_{e\in E(G_t)} m(e)} [\overline{\mathscr{M}}_{r(G_t)}].$$

□

Proposition 2.2.5. *Given graph cover G, let $H \twoheadrightarrow G_t$ be a contraction. Define the set*

$$S := \{G' | \exists f : G' \twoheadrightarrow G, G'_t = H\}.$$

Then we have

$$d_G = \sum_{G'\in S} d_{G'}.$$

Proof. Recall that d_G is the degree of the étale morphism $t : \overline{\mathscr{M}}_G \to \overline{\mathscr{M}}_{G_t}$ as in the proof of Proposition 3.3.2. Because t is étale , we have

$$t^*[\overline{\mathscr{M}}_H] = \sum_{G'\in S} [\overline{\mathscr{M}}_{G'}].$$

So the sum $\sum_{G'\in S} d_{G'}$ is the degree of t over $\overline{\mathscr{M}}_H \subset \overline{\mathscr{M}}_{G_t}$. □

2.3 Intersection theory on $\overline{\mathscr{M}}_\eta$

In this section we will review the definition of the Psi-classes and some results about the pullback of cycles on $\overline{\mathscr{M}}_\eta$ in [12]. Most importantly, given a graph cover H with H_t of genus 0, the intersection computation on $\overline{\mathscr{M}}_H$ can be transferred to $\overline{\mathscr{M}}_{r(H_t)}$, which is the moduli space of stable rational curves.

We recall the definition of Psi-classes from [12, Section 2.5]. Given a (twisted) graph G, let $l \in L(G)$ be one of the legs. There is a section

$$\sigma_l : \overline{\mathscr{M}}_G \to \mathfrak{C}_G,$$

where $\pi : \mathfrak{C}_G \to \overline{\mathscr{M}}_G$ is the universal curve. Define

$$L_l := \sigma_l^* \omega_\pi$$

to be the relative cotangent bundle which is the tautological line bundle. Given a graph cover H and $l \in L(H_s)$ (or $l \in L(H_t)$) the tautological line bundle L_l is the pullback from $\overline{\mathscr{M}}_{H_s}$ (or $\overline{\mathscr{M}}_{H_t}$). If l' maps to l in the graph cover, then $L_{l'} = L_l$.

Definition 2.3.1. The psi-class ψ_l associated to the leg l is

$$\psi_l := c_1(L_l).$$

Given a contraction $G \to H$, the moduli stack $\overline{\mathscr{M}}_G$ is a closed substack of $\overline{\mathscr{M}}_H$. There is a cycle $[\overline{\mathscr{M}}_G] \in A_*(\overline{\mathscr{M}}_H)$ by [19, Chapter 4].

Proposition 2.3.2. *The morphism $\overline{\mathscr{M}}_G \to \overline{\mathscr{M}}_{r(G_t)}$ is flat, proper and quasi-finite for a graph cover G with no edges in G_t.*

Proof. Assume G is a graph cover of degree d and $r(G_t)$ is of type g,n.

The morphism $\mathscr{B}^{\mathrm{bal}}_{g,n}(S_d) \to \overline{\mathscr{M}}_{g,n}$ is flat, proper and quasi-finite by [4, Corollary 3.0.5]. Here $\mathscr{B}^{\mathrm{bal}}_{g,n}(S_d)$ is the stack of balanced twisted stable maps of type g,n to $\mathrm{B}S_d$. We can associate an étale cover of degree d to each stable maps by Proposition 2.1.5. So by [12, Lemma 2.4.2],

$$\coprod_{H:r(H_t)=r(G_t)} \overline{\mathscr{M}}_H \cong \mathscr{B}^{\mathrm{bal}}_{g,n}(S_d),$$

i.e., the stack $\mathscr{B}^{\mathrm{bal}}_{g,n}(S_d)$ is the disjoint union of moduli stacks $\overline{\mathscr{M}}_H$ of all possible graph covers H. Then $\overline{\mathscr{M}}_H \to \overline{\mathscr{M}}_{g,n}$ must be flat, proper and quasi-finite for any H with $r(H_t)$ being of type g,n. □

Now we can pull back cycles from $\overline{\mathscr{M}}_{r(G_t)}$ to $\overline{\mathscr{M}}_G$ for a graph cover G.

Proposition 2.3.3. *Let η be a graph cover where η_t has 1 vertex and no edges. Assume $f : H \twoheadrightarrow r(\eta_t)$ satisfies $\#E(f) = 1$. Then*

$$t^* r^* [\overline{\mathscr{M}}_H] = \sum_{g:H' \twoheadrightarrow \eta} m(e) [\overline{\mathscr{M}}_{H'}],$$

where the sum is over $g : H' \twoheadrightarrow \eta$ such that $r(g) = f$, and $e \in E(g)$ is the unique element.

Proof. The pull back by r^* is given in [12, Lemma 4.0.2] where the factors $m(e)$ come from the fact that $r : \overline{\mathscr{M}}_{\eta_t} \to \overline{\mathscr{M}}_{r(\eta_t)}$ has ramification $x \mapsto x^{m(e)}$ along the divisor $\overline{\mathscr{M}}_G$ where e is the unique element in $E(G)$.

The pull back via t^* is straightforward because t is étale . □

The notation ψ_l is abused for simplicity. The ψ_l is either the psi-class associated to the leg $l \in L(G_t)$ or the psi-class associated to the image of l in $L(r(G_t))$, depending on where ψ_l sits as implied by the equations.

Proposition 2.3.4. *For the psi-class ψ_l, the pullback*

$$t^* r^* \psi_l = m(l) \psi_l.$$

Proof. The tautological line bundle satisfies $r^* L_l = L_l^{\otimes m(l)}$ [12, Section 2.5]. By definition, the tautological line bundle $t^* L_l = L_l$. So $t^* r^* \psi_l = m(l) \psi_l$. □

Given an edge $e = \{f_1, f_2\}$ of a twisted graph G where f_1, f_2 are the two flags of e, the tautological line bundle L_e on $\overline{\mathscr{M}}_G$ is the tensor $L_{f_1} \otimes L_{f_2}$. To be more precise, consider the twisted nodal curve

$$\mathscr{C}_U \to U$$

defining $U \to \overline{\mathscr{M}}_G$, locally the section given by the node e gives two tautological line bundles L_{f_1} and L_{f_2} on U. The line bundle L_e is then the tensor of L_{f_1} and L_{f_2}. This is the same as the L_e defined in [12, Chapter 4]. For graph cover H, if $e \in E(H_s)$ (or $e \in E(H_t)$), then L_e is the pull back from $\overline{\mathscr{M}}_{H_s}$(or $\overline{\mathscr{M}}_{H_t}$). The first Chern class of L_e is denoted by

$$\psi_e = c_1(L_e).$$

Suppose we have a diagram of contractions

$$\begin{tikzcd} & H'' \arrow[d, "g"] \\ H' \arrow[r, two heads, "f"] & H \end{tikzcd},$$

then $[\overline{\mathscr{M}}_{H''}]$ and $[\overline{\mathscr{M}}_{H'}]$ are cycles in $A_* \overline{\mathscr{M}}_H$. We want to calculate their intersections. For simplicity we assume $\#E(g) = 1$, i.e., the cycle $[\overline{\mathscr{M}}_{H''}]$ is a divisor.

Proposition 2.3.5. *If there exists a contraction $H' \twoheadrightarrow H''$, then $\overline{\mathscr{M}}_{H'} \subset \overline{\mathscr{M}}_{H''}$. The intersection*

$$[\overline{\mathscr{M}}_{H'}].[\overline{\mathscr{M}}_{H''}] = -\psi_e \smile [\overline{\mathscr{M}}_{H'}] \in A_* \overline{\mathscr{M}}_{H'}.$$

Otherwise,

$$[\overline{\mathscr{M}}_{H'}].[\overline{\mathscr{M}}_{H''}] = \sum_G [\overline{\mathscr{M}}_G]$$

where G satisfies $\#E(G_t) = \#E(H'_t) + 1$ *and makes the diagram*

$$\begin{tikzcd} G \arrow[r, two heads] \arrow[d, two heads] & H'' \arrow[d, two heads, "g"] \\ H' \arrow[r, two heads, "f"] & H \end{tikzcd}$$

commutes.

Proof. This is essentially [12, Lemma 4.0.1]. □

Chapter 3

Tropical intersection theory on $\mathscr{M}_\eta^{\mathrm{trop}}$

In this chapter, we study the ψ-classes and boundary divisors on the cone stacks $\mathscr{M}_\eta^{\mathrm{trop}}$ of twisted coverings. Recall that in order to talk about tropical cycles we first tropicalize $\overline{\mathscr{M}}_{0,n}$ to get the fan $\mathscr{M}_{0,n}^{\mathrm{trop}}$. Then $\mathscr{M}_{0,n}^{\mathrm{trop}}$ is embedded into $\mathbb{R}^{\binom{n}{2}-n}$ and viewed as subfan of Σ_n. We will tropicalize $\overline{\mathscr{M}}_\eta$ to get a cone stack $\mathscr{M}_\eta^{\mathrm{trop}}$. Instead of having an embedding into some $\mathbb{R}^N$, the cone stack $\mathscr{M}_\eta^{\mathrm{trop}}$ has a map to $\mathbb{R}^N$ which behaves like an étale morphism. This property makes intersection theory on $\mathscr{M}_\eta^{\mathrm{trop}}$ possible.

3.1 Tropicalization

In this section we will study the tropicalization of the moduli space of stable curves $\overline{\mathscr{M}}_{g,n}$ and of the moduli space of coverings of twisted stable curves $\overline{\mathscr{M}}_\eta$.

Tropicalizing $\overline{\mathscr{M}}_{0,n}$ helps to embed $\overline{\mathscr{M}}_{0,n}$ into a toric variety. Moreover, the tropicalization translates the usual intersection calculation on $\overline{\mathscr{M}}_{0,n}$ into tropical intersection calculation on $\mathscr{M}_{0,n}^{\mathrm{trop}}$.

There are different ways to tropicalize which leads to different tropicalizations. The tropicalization of $\overline{\mathscr{M}}_\eta$ via Berkovich space is studied in [10]. A set-theoretic tropicalization is studied in [1]. Tropicalization of $\overline{\mathscr{M}}_{g,n}$ via logarithmic structure can be done by putting log structure on $\overline{\mathscr{M}}_{g,n}$ [23].

Here we take the approach in [9] and define the tropicalization space as a cone stack. Note that $\mathscr{M}_{0,n}^{\mathrm{trop}}$ is not only the tropicalization of $\overline{\mathscr{M}}_{0,n}$ but also the moduli space of genus 0 stable tropical curves with n marked points. To generalize, we take the moduli space $\mathscr{M}_{g,n}^{\mathrm{trop}}$ of tropical stable curves and the moduli space $\mathscr{M}_\eta^{\mathrm{trop}}$ of type η coverings as the tropicalization of $\overline{\mathscr{M}}_{g,n}$ and $\overline{\mathscr{M}}_\eta$ respectively. The moduli space $\mathscr{M}_{g,n}^{\mathrm{trop}}$ is a cone stack [9]. We will review some of their results and apply them to define $\mathscr{M}_\eta^{\mathrm{trop}}$.

Let $\sigma \subset N_{\mathbb{R}}$ be a rational polyhedral cone with dual monoid $S_\sigma := \sigma^\vee \cap M$ where M is the dual of N. Denote $S_\sigma \backslash \{0\}$ by S_σ^*.

Definition 3.1.1. [9, Definition 3.2] A *tropical curve* $\Gamma = (G, d)$ over σ is a connected weighted marked graph G with a function $d : E(G) \to S_\sigma^*$.

The function d is called the *metric* on Γ. A tropical curve is of genus g with n marked points if the genus of G is g and $L(G)$ is marked by $[n]$. A tropical curve is *stable* if the underlying weighted marked graph is stable. When there is no ambiguity, we simply write Γ for a tropical curve. The underlying weighted marked graph is denoted by $\mathbb{G}(\Gamma)$.

A morphism $f : \Gamma_1 \to \Gamma_2$ of tropical curves over σ_1 and σ_2 respectively is an isomorphism $\mathbb{G}(f) : \mathbb{G}(\Gamma_1) \to \mathbb{G}(\Gamma_2)$ of marked weighted graphs preserving weighting and marking and a morphism $f_\sigma : \sigma_1 \to \sigma_2$ of rational polyhedral cones such that for each edge e in $\mathbb{G}(\Gamma_1)$, the edge length $d_1(e) = f_\sigma^\vee \circ d_2 \circ \mathbb{G}(f)(e)$, where $f_\sigma^\vee : \sigma_2^\vee \to \sigma_1^\vee$ is the dual function of f_σ.

Definition 3.1.2. [9, Section 2.1] The category $\mathbf{RPC}_{\mathbb{Z}}$ is the category of rational polyhedral cones (N, σ) such that $S_\sigma = \sigma^\vee \cap M$ is sharp with morphisms being $\mathbb{Z}$-linear morphisms of rational polyhedral cones.

A morphism $\tau \to \sigma$ in $\mathbf{RPC}_{\mathbb{Z}}$ is said to be a *face morphism* if it induces an isomorphism of τ onto a face of σ.

Definition 3.1.3. [9, Definition 2.1] A *(rational polyhedral) cone complex* Σ is a collection of cones $\mathscr{C} = \{\sigma_\alpha\}$ in $\mathbf{RPC}_{\mathbb{Z}}$ together with a collection of face morphisms $\mathscr{F} = \{\phi_{\alpha\beta} : \sigma_\alpha \to \sigma_\beta\}$ closed under composition such that the following three axioms are fulfilled:

(i) The identity map $\mathrm{id}_\alpha : \sigma_\alpha \to \sigma_\alpha$ is in $\mathscr{F}$.

(ii) Every face of a cone σ_β is the image of exactly one face morphism $\phi_{\alpha\beta}$ in $\mathscr{F}$.

(iii) There is at most one morphism $\sigma_\alpha \to \sigma_\beta$ between two cones σ_α and σ_β in $\mathscr{F}$.

Denote the category of (rational polyhedral) cone complex by $\mathbf{RPCC}_{\mathbb{Z}}$. This is equivalent to the category of *rational polyhedral cone complexes* in [1]. The collection $\mathscr{F}$ of face morphisms tells whether σ_α is a face of σ_β.

The *morphism* $f : \Sigma_1 \to \Sigma_2$ in $\mathbf{RPCC}_{\mathbb{Z}}$ is given by choosing a morphism $f_{\alpha_1\alpha_2} : \sigma_{\alpha_1} \to \sigma_{\alpha_2}$ on each cone in Σ_1 which does not factor through any proper face τ of σ_{α_2} and compatible with the collections $\mathscr{F}_1$ and $\mathscr{F}_2$. The compatibility means that for every $\phi_{\alpha_1\beta_1} \in \mathscr{F}_1$, there exists $\phi_{\alpha_2\beta_2} : \sigma_{\alpha_2} \to \sigma_{\beta_2} \in \mathscr{F}_2$ making the diagram

$$\begin{tikzcd} \sigma_{\alpha_1} \arrow[r, "f_{\alpha_1\alpha_2}"] \arrow[d, "\phi_{\alpha_1\beta_1}"'] & \sigma_{\alpha_2} \arrow[d, "\phi_{\alpha_2\beta_2}"] \\ \sigma_{\beta_1} \arrow[r, "f_{\beta_1\beta_2}"] & \sigma_{\beta_2} \end{tikzcd}$$

commute.

A morphism $f : \Sigma_1 \to \Sigma_2$ of cone complexes is called *strict* if the morphisms $f_{\alpha_1\alpha_2} : \sigma_{\alpha_1} \to \sigma_{\alpha_2}$ are all isomorphisms.

Remark 3.1.4. Roughly speaking, the face morphisms are like étale morphisms while the strict morphisms are like smooth morphisms in the familiar theory of algebraic spaces and algebraic stacks. So similar to the étale site, there is τ_{face} site on $\mathbf{RPCC}_{\mathbb{Z}}$. The theory for *cone stacks* and *cone spaces* is in [9, Chapter 2]. For more detail, see e.g. [43, Section 1.3.2].

Fix a stable weighted marked graph G, consider the functor

$$U_G : \mathbf{RPC}_{\mathbb{Z}}^{\mathrm{op}} \to \mathbf{Groupoids}$$

that associates to a rational polyhedral cone σ the groupoid whose objects are pairs (Γ, ϕ) consisting of a tropical curve Γ over σ and a contraction $\phi : G \twoheadrightarrow \mathbb{G}(\Gamma)$. The functor U_G determines a stack over $(\mathbf{RPCC}_{\mathbb{Z}}, \tau_{face})$, also denoted by U_G, whose fiber over a rational polyhedral cone σ is $U_G(\sigma)$.

Proposition 3.1.5. *[9, Lemma 3.4] The stack U_G is represented by the rational polyhedral cone $\sigma_G = \mathbb{R}_{\geq 0}^{E(G)}$.*

Definition 3.1.6. [9, Definition 3.3] The stack $\mathscr{M}_{g,n}^{\mathrm{trop}}$ is the stack over $\mathbf{RPCC}_{\mathbb{Z}}$ whose fiber over a cone σ is the groupoid of genus g, n-marked stable tropical curves over σ.

Here we only specify the fiber over all rational polyhedral cones. The stack $\mathscr{M}_{g,n}^{\mathrm{trop}}$ over $\mathbf{RPCC}_{\mathbb{Z}}$ is well-defined by Proposition 3.1.7.

Proposition 3.1.7. *[9, Proposition 2.3] The 2-category of stacks on $\mathbf{RPCC}_{\mathbb{Z}}$ is equivalent to the 2-category of categories fibered in groupoids on $\mathbf{RPC}_{\mathbb{Z}}$ via the natural restriction.*

Definition 3.1.8. A *cone space* is a sheaf $X : \mathbf{RPCC}_{\mathbb{Z}} \to \mathbf{Sets}$ that fulfills the following conditions:

- The diagonal morphism $\Delta : X \to X \times X$ is representable by cone complexes in $\mathbf{RPCC}_{\mathbb{Z}}$.

- There is a representable strict morphism $U \to X$ from a cone complex U that is surjective.

Definition 3.1.9. A *cone stack* is a stack $\mathscr{X}$ over $\mathbf{RPCC}_{\mathbb{Z}}$ that fulfills the following conditions:

- The diagonal morphism $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable by cone spaces.
- There is a representable strict morphism $\mathscr{U} \to \mathscr{X}$ from a geometric space $\mathscr{U}$ that is surjective.

Proposition 3.1.10. *[9, Theorem 1] The stack $\mathscr{M}_{g,n}^{trop}$ is a cone stack.*

Here is the groupoid representation for $\mathscr{M}_{g,n}^{\mathrm{trop}}$ from [9, Section 3.3]. By definition, the stack $\mathscr{M}_{g,n}^{\mathrm{trop}}$ is covered by $\mathscr{U} = \coprod_G U_G$, where G runs over all genus g, n-marked graphs. Moreover, the cover $\mathscr{U} \to \mathscr{M}_{g,n}^{\mathrm{trop}}$ is strict and representable[9, Lemma 3.5]. So there is a groupoid presentation of $\mathscr{M}_{g,n}^{\mathrm{trop}} \cong (\mathscr{R} \rightrightarrows \mathscr{U})$, where $\mathscr{R} = \mathscr{U} \times_{\mathscr{M}_{g,n}^{\mathrm{trop}}} \mathscr{U}$. The space $\mathscr{R}$ decomposes as a disjoint union

$$\mathscr{R} = \coprod_{G_1,G_2} R_{G_1,G_2}, \text{ where } R_{G_1,G_2} = U_{G_1} \times_{\mathscr{M}_{g,n}^{\mathrm{trop}}} U_{G_2}.$$

The space R_{G_1,G_2} is represented by a cone complex consisting of $C(S_1,S_2,\phi) \cong \mathbb{R}_{\geq 0}^{E(G_1/S_1)}$ indexed by $(S_1 \subset E(G_1), S_2 \subset E(G_2), \phi : G_1/S_1 \xrightarrow{\cong} G_2/S_2)$, where G_i/S_i is the graph G_i contracting its edges S_i.

Proposition 3.1.11. *The cone stack $\mathscr{M}_{0,n}^{trop}$ is a cone complex. Moreover, the cone complex $\mathscr{M}_{0,n}^{trop}$ consists of all the cones U_H for type $0,n$ weighted marked graph H with U_H being a face of U_G if and only if there is a contraction $G \twoheadrightarrow H$.*

Proof. Consider the relation R_{G_1,G_2} consisting of the cones $C(S_1,S_2,\phi)$. Notice that given G_i and S_i, there is at most one ϕ because the genus is 0. So there is no orbifold structure on $\mathscr{M}_{0,n}^{\mathrm{trop}}$. Hence $\mathscr{M}_{0,n}^{\mathrm{trop}}$ is a cone complex. The statement about the face morphisms follows from the groupoid representation. $\square$

We will generalize this definition to twisted tropical curves. We will also define tropical étale covers and discuss the moduli stack of the covers.

Definition 3.1.12. Given a rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, a *twisted tropical curve* $\Gamma = (G,d)$ over σ is a twisted graph G with a function $d : E(G) \to S_{\sigma}^{*}$.

The function $d : E(G) \to S_\sigma^*$ gives the nonzero edge lengths on $E(G)$. Given a twisted tropical curve Γ, we write $\mathbb{G}(\Gamma)$ for its underlying twisted graph.

A type η of twisted graph is a twisted graph with 1 vertex but no edges. A twisted tropical curve Γ is of type η if there is a contraction $\mathbb{G}(\Gamma) \twoheadrightarrow \eta$. We can simply contract all the edges in $E(\mathbb{G}(\Gamma))$ to get the type of Γ. We say a twisted tropical curve Γ is stable if $\mathbb{G}(\Gamma)$ is.

A morphism of twisted tropical curves $(\Gamma, \sigma) \to (\Gamma', \sigma')$ is a morphism $f : \sigma \to \sigma'$ in $\mathbf{RPC}_\mathbb{Z}$ and an isomorphism of their twisted graphs $g : \mathbb{G}(\Gamma) \to \mathbb{G}(\Gamma')$ such that for each edge e of $\mathbb{G}(\Gamma)$ we have $d(e) = f^\vee \circ d' \circ g(e)$.

The *coarse tropical curve* $r(\Gamma)$ of the twisted tropical curve $\Gamma = (G, d)$ is the tropical curve $(r(G), d)$.

Definition 3.1.13. The stack $\mathscr{M}_\eta^{\mathrm{trop}}$ of twisted tropical curves is the stack over $\mathbf{RPCC}_\mathbb{Z}$ whose fiber over σ is the groupoid of stable type η twisted tropical curves over σ.

Again the stack $\mathscr{M}_\eta^{\mathrm{trop}}$ is well defined by Proposition 3.1.7.

Definition 3.1.14. A *tropical étale cover* $\Gamma = (G, d_t)$ over σ is a graph cover $G = G_s \to G_t$ with a function $d_t : E(G_t) \to S_\sigma^*$.

The metric d_t gives the edge lengths on the target twisted graph G_t. We have a natural induced metric d_s on the source twisted graph by composition. So a tropical étale cover can indeed be viewed as a covering between two twisted tropical curves where the metric on the source twisted tropical curve is induced from that on the target twisted tropical curve.

A type $\eta = (\pi : \eta_s \to \eta_t)$ of a tropical étale cover is a graph cover with 1 vertex in the target twisted graph and no edges. The types of the target and the source twisted graphs of a tropical étale cover of type η are given by η_s and η_t respectively. In particular, the map π specifies how the legs of the source twisted graph map to the legs of the target twisted graph but have no control over the edges. A tropical étale cover Γ is of type η if its underlying graph cover $\mathbb{G}(\Gamma)$ is, i.e., there is a contraction $\mathbb{G}(\Gamma) \to \eta$.

Given a tropical étale cover $\Gamma = (G, d_t)$, the source twisted tropical curve is (G_s, d_s) and the target twisted tropical curve is (G_t, d_t). A tropical étale cover is stable if its target twisted tropical curve is stable. This implies that the source twisted tropical curve is stable.

Given a morphism $f : \tau \to \sigma$ and a tropical étale cover $\Gamma_\sigma = (G, d)$ over σ, we define the tropical étale cover

$$f^*\Gamma_\sigma := (H, d_H)$$

where $d_H = f^\vee \circ d$ and H is G contracting all edges mapping to 0 by d_H. If f is a face morphism, we denote $f^*\Gamma_\sigma$ by $\Gamma_\sigma|_\tau$.

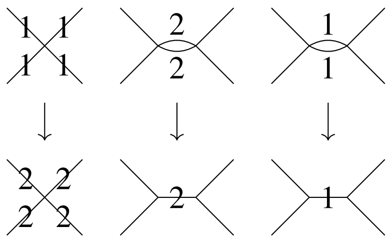
Definition 3.1.15. The stack $\mathscr{M}_\eta^{\mathrm{trop}}$ of tropical étale covers is the stack over $\mathbf{RPCC}_\mathbb{Z}$ whose fiber over σ is the groupoid of stable type η tropical étale covers over σ.

Recall by definition of graph cover, the type $\eta = (\pi : \eta_s \to \eta_t)$ must satisfy

- (Degree) The degree $d = \sum_{i_s \in \pi^{-1}(i_t)} \frac{m_t(i_t)}{m_s(i_s)}$ is invariant for all $i_t \in L(\eta_t)$.
- (Hurwitz Formula) $2g_s - 2 = d(2g_t - 2) + \sum_{i_s \in \pi^{-1}(i_t)} (\frac{m_t(i_t)}{m_s(i_s)} - 1)$.

On a covering of twisted graphs, the type η has specified the images and the degrees of legs but not of edges. So a covering G of type η may not be a graph cover because G may not be representable.

Example 3.1.16. Here are some maps between twisted graphs with the values of m on their legs. All three maps have the same type η. But the middle one is not representable by Proposition 2.1.7, hence not a graph cover.



The legs of the second and third pictures have the same t as shown in the first picture. The vertex in the source of the first picture has genus 1. All other vertices has genus 0.

After some preparation, we will prove the stack $\mathscr{M}_\eta^{\mathrm{trop}}$ is a cone stack for a twisted tropical curves covering type η, i.e.

- The diagonal map $\Delta : \mathscr{M}_\eta^{\mathrm{trop}} \to \mathscr{M}_\eta^{\mathrm{trop}} \times \mathscr{M}_\eta^{\mathrm{trop}}$ is representable by cone spaces.
- There is a morphism $\mathscr{U} \to \mathscr{M}_\eta^{\mathrm{trop}}$ from a cone complex $\mathscr{U}$ that is surjective and strict.

Fix a graph cover $G = (G_s, G_t, \pi : G_s \to G_t)$ of type η. Consider the functor

$$U_G : \mathbf{RPC}_\mathbb{Z}^{\mathrm{op}} \to \mathbf{Groupoids}$$

that associates to a rational polyhedral cone σ the groupoid whose objects are pairs (Γ, ϕ) where

- the tropical étale cover Γ over σ is of type η;

- the morphism ϕ is a contraction $\phi : G \twoheadrightarrow \mathbb{G}(\Gamma)$ of graph covers.

Proposition 3.1.17. *The stack U_G is represented by the rational polyhedral cone $\sigma_G = \mathbb{R}_{\geq 0}^{E(G_t)}$.*

Proof. Let σ be a rational polyhedral cone.

An automorphism f of Γ in the groupoid $U_G(\sigma)$ must be the identity. Indeed, the automorphism f makes the diagram

$$\begin{tikzcd} \mathbb{G}(\Gamma) \arrow[rr, "f"] & & \mathbb{G}(\Gamma) \\ & G \arrow[ul, two heads, "\phi"] \arrow[ur, two heads, "\phi"'] & \end{tikzcd}$$

commute. Note that G and $\mathbb{G}(\Gamma)$ are both of type η. The morphism ϕ is a contraction of edges, hence surjective. So f has to be the identity.

Then it suffices to prove there is a set-wise natural bijection

$$\mathrm{Hom}(\sigma, \sigma_G) \xrightarrow{\cong} U_G(\sigma).$$

Indeed, the left hand side is $\mathrm{Hom}(\sigma, \mathbb{R}_{\geq 0}^{E(G_t)}) = \mathrm{Hom}(\sigma, \mathbb{R}_{\geq 0})^{E(G_t)}$. The right hand side is $S_\sigma^{E(G_t)}$. Remember the morphisms of polyhedral cones in $\mathrm{Hom}(\sigma, \mathbb{R}_{\geq 0})$ preserve the lattice structure. The natural bijection is given by the bijection $\mathrm{Hom}(\sigma, \mathbb{R}_{\geq 0}) = \sigma^\vee \cap M$. $\square$

Proposition 3.1.18. *The morphism $U_G \to \mathscr{M}_\eta^{trop}$ is representable by cone spaces, strict, and quasicompact.*

Proof. Let $\sigma \to \mathscr{M}_\eta^{\mathrm{trop}}$ be a morphism from an object σ in $\mathbf{RPC}_\mathbb{Z}$. It suffices to show that the fiber product

$$X_{\sigma,G} = U_G \times_{\mathscr{M}_\eta^{\mathrm{trop}}} \sigma$$

is representable by a cone complex with finitely many cones and that the induced map $X_{\sigma,G} \to \sigma$ is strict.

Let the morphism $\sigma \to \mathscr{M}_\eta^{\mathrm{trop}}$ be given by the tropical étale cover Γ_σ over σ. By the definition of fiber product of stacks, the objects in the groupoid $X_{\sigma,G}(\alpha)$ over the object α in $\mathbf{RPC}_\mathbb{Z}$ are the triples

$$\{(i : \alpha \to \sigma, (\Gamma_\alpha, \phi_\alpha) \in U_G, \Gamma_\alpha \xrightarrow{\cong} i^*\Gamma_\sigma)\}.$$

Here the morphism $\alpha \to U_G$ gives the tropical étale cover Γ_α over α and $\phi_\alpha : G \twoheadrightarrow \mathbb{G}(\Gamma_\alpha)$. The data can be pared down to

$$\{(i : \alpha \to \sigma, G \twoheadrightarrow \mathbb{G}(i^*\Gamma_\sigma))\}.$$

Consider the factorization of the map $i : \alpha \to \sigma$ through some face τ of σ such that $\alpha \to \tau$ does not factor through any proper face of τ. The fibre product $X_{\sigma,G}$ is represented by the cone complex consisting of $C(\tau,\phi) \cong \tau$ indexed by pairs

$$(j : \tau \to \sigma, \phi : G \twoheadrightarrow \mathbb{G}(j^*\Gamma_\sigma)),$$

where $j : \tau \to \sigma$ is a face morphism, i.e., the fibre product $X_{\sigma,G}$ is a collection of copies of the faces of σ indexed by the face τ and a morphism $\phi : G \twoheadrightarrow \mathbb{G}(\Gamma_\sigma|_\tau)$. The face morphisms on $X_{\sigma,G}$ is given by $C(\tau',\phi') \le C(\tau,\phi)$ if and only if $\tau' \le \tau$ and the diagram

$$\begin{tikzcd} \mathbb{G}(\Gamma_\sigma|_\tau) \arrow[rr, two heads] & & \mathbb{G}(\Gamma_\sigma|_{\tau'}) \\ & \Gamma \arrow[ul, two heads, "\phi"] \arrow[ur, two heads, "\phi'"'] & \end{tikzcd}$$

is commutative. So $U_G \to \mathscr{M}_\eta^{\text{trop}}$ is representable.

The induced map $X_{\sigma,G} \to \sigma$ is mapping $C(\tau,\phi)$ to τ, hence strict.

The cone complex $X_{\sigma,G}$ consists of finite cones. Indeed, there are only finitely many choices of faces τ of σ. There are finitely many choices of $\mathbb{G}(\Gamma_\sigma|_\tau)$ because of the representability condition of the graph covers, hence finitely many choices of contractions ϕ. $\square$

Theorem 3.1.19. *The moduli functor $\mathscr{M}_\eta^{trop}$ of tropical étale covers is representable by a cone stack.*

Proof. Let $\mathscr{U}$ be the disjoint union of all U_G where G is of type η. Then the morphism $\mathscr{U} \to \mathscr{M}_\eta^{\text{trop}}$ is still representable by cone spaces, strict, and quasicompact.

Moreover, this morphism is surjective. Indeed, consider the Cartesian diagram

$$\begin{tikzcd} \coprod_G X_{\sigma,G} \arrow[r] \arrow[d] & \mathscr{U} \arrow[d] \\ \sigma \arrow[r] & \mathscr{M}_\eta^{\text{trop}} \end{tikzcd}$$

and assume $\sigma \to \mathscr{M}_\eta^{\mathrm{trop}}$ is given by the tropical étale cover Γ_σ. In the proof of Proposition 3.1.18, we show that $C(\sigma, id) \cong \sigma$ is a face of $X_{\sigma,\mathbb{G}(\Gamma_\sigma)}$. Moreover, the cone $C(\sigma, id)$ maps isomorphically to σ. So $\mathscr{U} \to \mathscr{M}_\eta^{\mathrm{trop}}$ is surjective.

We only need to show $\Delta : \mathscr{M}_\eta^{\mathrm{trop}} \to \mathscr{M}_\eta^{\mathrm{trop}} \times \mathscr{M}_\eta^{\mathrm{trop}}$ is representable by a cone space.

The fibre product $\mathscr{U} \times_{\mathscr{M}_\eta^{\mathrm{trop}}} \mathscr{U}$ is a cone space. Indeed, a base change $\mathscr{U} \times_{\mathscr{M}_\eta^{\mathrm{trop}}} \mathscr{U} \to \mathscr{U}$ of the representable morphism $\mathscr{U} \to \mathscr{M}_\eta^{\mathrm{trop}}$ is again representable. Notice that our $\mathscr{U}$ is a cone complex. So $\mathscr{U} \times_{\mathscr{M}_\eta^{\mathrm{trop}}} \mathscr{U}$ is a cone space.

We have another diagram

$$\begin{tikzcd}
\mathscr{U} \times_{\mathscr{M}_\eta^{\mathrm{trop}}} \mathscr{U} \arrow[r] \arrow[d] & \mathscr{M}_\eta^{\mathrm{trop}} \arrow[d, "\Delta"] \\
\mathscr{U} \times \mathscr{U} \arrow[r] & \mathscr{M}_\eta^{\mathrm{trop}} \times \mathscr{M}_\eta^{\mathrm{trop}}
\end{tikzcd}$$

which is Cartesian. Any morphism from a cone $\sigma \to \mathscr{M}_\eta^{\mathrm{trop}} \times \mathscr{M}_\eta^{\mathrm{trop}}$ will factor through $\mathscr{U} \times \mathscr{U}$ because $\mathscr{U} \times \mathscr{U} \to \mathscr{M}_\eta^{\mathrm{trop}} \times \mathscr{M}_\eta^{\mathrm{trop}}$ is strict and surjective. Hence

$$\sigma \times_{\mathscr{M}_\eta^{\mathrm{trop}} \times \mathscr{M}_\eta^{\mathrm{trop}}} \mathscr{M}_\eta^{\mathrm{trop}} = \sigma \times_{\mathscr{U} \times \mathscr{U}} (\mathscr{U} \times_{\mathscr{M}_\eta^{\mathrm{trop}}} \mathscr{U})$$

is a cone space because σ, $\mathscr{U} \times \mathscr{U}$ and $\mathscr{U} \times_{\mathscr{M}_\eta^{\mathrm{trop}}} \mathscr{U}$ are all cone spaces. So the morphism Δ is representable by a cone space. $\square$

The fan structure on $\mathscr{M}_\eta^{\mathrm{trop}}$ is given by the morphisms of type η graph covers. Given a graph cover H of type η, define

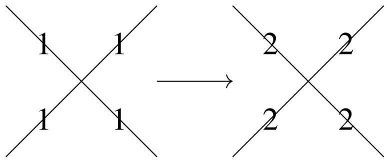
$$R_H := U_H \times_{\mathscr{M}_\eta^{\mathrm{trop}}} U_H$$

and

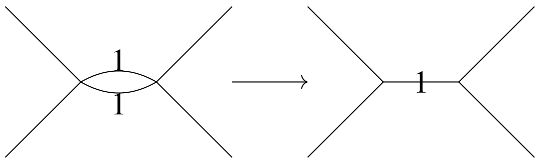
$$\mathscr{M}_H^{\mathrm{trop}} := (R_H \rightrightarrows U_H).$$

The cone stack $\mathscr{M}_H^{\mathrm{trop}}$ is a sub cone stack of $\mathscr{M}_\eta^{\mathrm{trop}}$. The dimension of the cone stack $\mathscr{M}_H^{\mathrm{trop}}$ is the same as the dimension of U_H because the automorphism group is always finite. We call $\mathscr{M}_H^{\mathrm{trop}}$ a stratum of $\mathscr{M}_\eta^{\mathrm{trop}}$. For two graph covers H_1 and H_2 of type η, the cone stack $\mathscr{M}_{H_1}^{\mathrm{trop}}$ is a face of $\mathscr{M}_{H_2}^{\mathrm{trop}}$ if and only if there is a contraction $H_1 \twoheadrightarrow H_2$ of graph covers.

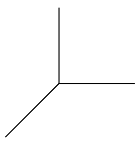
Example 3.1.20. Let type η be the graph cover from genus 1, 4-legged graph to genus 0, 4-legged graph with multiplicity specified as follows.



Then the cone stack $\mathscr{M}_{\eta}^{\mathrm{trop}}$ consists of three one-dimensional rays with identity $\mathbb{Z}_2$ action. Each one-dimensional ray corresponds to a type η graph cover H with extra multiplicity on edges specified like the following.



The identity $\mathbb{Z}_2$ action comes from mapping the two edges in H_s to each other. There are 3 graph covers of this kind by mutation on the leg labelling, hence three rays. The point these three rays emitting from is the only 0-dimensional stratum corresponding to the graph cover η itself. So the coarse moduli space of $\mathscr{M}_{\eta}^{\mathrm{trop}}$ will look like the following.



3.2 Intersection theory

In this section we will discuss the tropical intersection theory on $\mathscr{M}_{\eta}^{\mathrm{trop}}$ where η is a graph cover whose target twisted graph η_t is of genus 0 with no edges.

While $\mathscr{M}_{\eta}^{\mathrm{trop}}$ generally does not embed into $\mathbb{R}^N$, the existence of a natural map $\mathscr{M}_{\eta}^{\mathrm{trop}} \to \mathscr{M}_{0,n}^{\mathrm{trop}}$ enables us to do intersection theory on $\mathscr{M}_{\eta}^{\mathrm{trop}}$.

A *k-dimensional weighted fan* (X, ω_X) of the cone stack $\mathscr{M}_{\eta}^{\mathrm{trop}}$ is a collection of k-dimensional strata X with weighted function $\omega_X : X \to \mathbb{Q}$ on each stratum in the collection. Recall a k-dimensional stratum on $\mathscr{M}_{\eta}^{\mathrm{trop}}$ is $\mathscr{M}_{H}^{\mathrm{trop}}$ for some graph cover H where $\#E(H_t) = k$.

Define the morphism $\mathrm{tc} : \mathscr{M}_{\eta}^{\mathrm{trop}} \to \mathscr{M}_{0,n}^{\mathrm{trop}}$ mapping a tropical étale cover to the coarse tropical curve of its target twisted tropical curve, i.e., the map $\mathrm{tc}(\sigma) : \mathscr{M}_{\eta}^{\mathrm{trop}}(\sigma) \to \mathscr{M}_{0,n}^{\mathrm{trop}}(\sigma)$

over σ is defined as

$$(G, d_G) \mapsto (r(G_t), d_G).$$

Recall we have the covering map $\coprod_G U_G \to \mathscr{M}_{0,n}^{\mathrm{trop}}$ and $\coprod_H U_H \to \mathscr{M}_\eta^{\mathrm{trop}}$ where G runs over all type $0,n$ weighted marked graph while H runs over all graph cover of type η.

Fix a weighted marked graph G of type $0,n$. Then there is the commutative diagram

$$\begin{tikzcd} \coprod_{H:r(H_t)=G} U_H \arrow[r] \arrow[d] & \mathscr{M}_\eta^{\mathrm{trop}} \arrow[d] \\ U_G \arrow[r] & \mathscr{M}_{0,n}^{\mathrm{trop}}. \end{tikzcd}$$

Note $U_G = \mathbb{R}_{\geq 0}^{E(G)}$ and $U_H = \mathbb{R}_{\geq 0}^{E(H_t)}$. The map $U_H \to U_G$ is the isomorphism by identifying $E(G) = E(H_t)$.

Remark 3.2.1. The coarse moduli cone of the subcone stack $\mathscr{M}_H^{\mathrm{trop}}$ is $\mathbb{R}_{\geq 0}^{E(H_t)}$. Indeed, notice that any automorphism of tropical étale cover G of type η is the identity on G_t when η_t is of genus 0. Moreover, on each cone stack $\mathscr{M}_H^{\mathrm{trop}}$, the morphism tc is simply forgetting the orbifold structure as any point is only identified with itself under the automorphisms. The morphism tc is like taking the coarse moduli space locally.

Example 3.2.2. We will investigate the map tc for the following η. Let η be the graph cover where the twisted graph η_s is 1 vertex of genus 1 with 4 legs of twisted degree 1; the twisted graph η_t is 1 vertex of genus 0 with 4 legs of twisted degree 2; the map of legs is 1 to 1. See the first graph cover in Example 3.1.16. Then $\mathscr{M}_\eta^{\mathrm{trop}}$ consists of the 0-dimensional cone $\{0\}$ and three 1-dimensional cone $[\mathbb{R}_{\geq 0}/\mathbb{Z}_2]$ where the group $\mathbb{Z}_2$ comes from swapping of the two bounded edges and both elements of $\mathbb{Z}_2$ acts trivially on $\mathbb{R}_{\geq 0}$.

Then tc maps $\mathscr{M}_\eta^{\mathrm{trop}}$ to $\mathscr{M}_{0,4}^{\mathrm{trop}}$ which consists of $\{0\}$ and three 1-dimensional cone $\mathbb{R}_{\geq 0}$. The map on the 0-dimensional cone is mapping $\{0\}$ to $\{0\}$ while the map on any of the 1-dimensional cone is

$$[\mathbb{R}_{\geq 0}/\mathbb{Z}_2] \to \mathbb{R}_{\geq 0}$$

which is exactly taking the coarse moduli space.

The map $\phi : \mathscr{M}_\eta^{\mathrm{trop}} \to \mathbb{R}^{\binom{n}{2}-n}$ is defined as the composition of $\mathrm{tc} : \mathscr{M}_\eta^{\mathrm{trop}} \to \mathscr{M}_{0,n}^{\mathrm{trop}}$ and the embedding $\mathscr{M}_{0,n}^{\mathrm{trop}} \to \mathbb{R}^{\binom{n}{2}-n}$. This map preserves the fan structure, i.e., if τ is a face of σ in $\mathscr{M}_\eta^{\mathrm{trop}}$, then $\phi(\tau)$ is a face of $\phi(\sigma)$. The primitive normal vector $u_{\sigma/\tau}$ is defined to be the primitive normal vector $u_{\phi(\sigma)/\phi(\tau)}$ for τ being a face of σ of codimension 1.

Recall that $\mathscr{M}_{0,n}^{\mathrm{trop}}$ consists of cones U_G for all type $0,n$ weighted marked graph G by Proposition 3.1.11. For graph cover H and K of type η, the primitive normal vector $u_{\mathscr{M}_H^{\mathrm{trop}}/\mathscr{M}_K^{\mathrm{trop}}}$ is exactly $u_{U_{r(H_t)}/U_{r(K_t)}}$, where $\mathscr{M}_H^{\mathrm{trop}}$ and $\mathscr{M}_K^{\mathrm{trop}}$ are cones of $\mathscr{M}_\eta^{\mathrm{trop}}$ while $U_{r(H_t)}$ and $U_{r(K_t)}$ are cones of $\mathscr{M}_{0,n}^{\mathrm{trop}}$. We define

$$u_{H/K} := u_{\mathscr{M}_H^{\mathrm{trop}}/\mathscr{M}_K^{\mathrm{trop}}}.$$

For $u_{H/K}$ to make sense, there has to exists a contraction $g : H \twoheadrightarrow K$ such that $\#E(g) = 1$.

A *k-dimensional tropical fan* on the cone stack $\mathscr{M}_\eta^{\mathrm{trop}}$ is a k-dimensional weighted fan (X,ω) whose weights satisfy the balancing condition, i.e., for any $(k-1)$-dimensional stratum $\mathscr{M}_H^{\mathrm{trop}}$,

$$\sum_{G_i:\exists G_i\twoheadrightarrow H} \omega(G_i)m(e_i)u_{G_i/H} = 0 \in \mathbb{R}^{\binom{n}{2}-n}/V_H,$$

where V_H are the subspace generated by $\phi(\mathscr{M}_H^{\mathrm{trop}})$; the weights $\omega(G_i)$ are short for the weight $\omega(\mathscr{M}_{G_i}^{\mathrm{trop}})$; the multiplicity $m(e_i)$ is the multiplicity of the the unique contracted edge e_i in the target map $G_{it} \twoheadrightarrow H_t$ of $G_i \twoheadrightarrow H$. The multiplicity term $m(e_i)$ arises from the pullback formula [12, Lemma 4.0.2]. Given a Chow cohomology class c of $\overline{\mathscr{M}}_\eta$, the weighted fan on the cone stack $\mathscr{M}_\eta^{\mathrm{trop}}$ with weights $c(\overline{\mathscr{M}}_H)$ on $\mathscr{M}_H^{\mathrm{trop}}$ is a tropical fan, following from the same proof of Proposition 1.1.5.

A *Cartier divisor* of the cone stack $\mathscr{M}_\eta^{\mathrm{trop}}$ is a piecewise linear function $f : \mathscr{M}_\eta^{\mathrm{trop}} \to \mathbb{R}$. Recall the coarse moduli cone of subcone stack $\mathscr{M}_H^{\mathrm{trop}}$ is $\mathbb{R}_{\geq 0}^{E(H_t)}$. Being piecewise linear simply means f is piecewise linear on the coarse moduli cone. See Remark 3.2.1.

The intersection of f and a k-dimensional tropical fan (X,ω_X) is the $(k-1)$-dimensional tropical fan whose weight ω on a $(k-1)$-dimensional stratum τ is

$$\omega(H) = \sum_{G_i:\exists G_i\twoheadrightarrow H} f_{G_i}(\omega_X(G_i)m(e_i)u_{G_i/H}) - f_H(\sum_{G_i:\exists G_i\twoheadrightarrow H} \omega_X(G_i)m(e_i)u_{G_i/H}),$$

where f_{G_i} and f_H are defined as the linear extension of the pushdown of $f|_{\mathscr{M}_{G_i}^{\mathrm{trop}}}$ and $f|_{\mathscr{M}_H^{\mathrm{trop}}}$ to $\mathbb{R}^{\binom{n}{2}-n}$ respectively. Recalling that $\mathscr{M}_H^{\mathrm{trop}} = (R_H \rightrightarrows U_H)$ and $U_H \xrightarrow{\cong} U_{r(H_t)} \hookrightarrow \mathbb{R}^{\binom{n}{2}-n}$, the pushdown of $f|_{\mathscr{M}_H^{\mathrm{trop}}}$ is a linear function on $U_{r(H_t)}$ whose linear extension will then be the linear function f_H on $\mathbb{R}^{\binom{n}{2}-n}$. We will denote the tropical fan by X and the intersection by $f \smile X$ when there is no ambiguity. If X is a tropical fan associated to the cohomology class c, then $f \smile X$ will be the tropical fan associated to the intersection of the Cartier divisor associated to f and c, following from the same proof of Proposition 1.1.8.

The tropical fan associated to a Chow cohomology class $[h] \in A^k(\overline{\mathscr{M}}_\eta)$ is the collection of $(n-3-k)$-dimensional strata $\mathscr{M}_G^{\text{trop}}$ with weights

$$\int_{\overline{\mathscr{M}}_G} [h] = [\overline{\mathscr{M}}_G].[h] \in A_0(\overline{\mathscr{M}}_\eta).$$

The fundamental tropical fan $[\mathscr{M}_\eta^{\text{trop}}]$ is the tropical fan associated to $1 \in A^0(\overline{\mathscr{M}}_\eta)$, i.e., collection of all top dimensional strata $\mathscr{M}_G^{\text{trop}}$ with weights

$$\omega(G) = \frac{d_G}{\prod_{e\in E(G_t)} m(e) \cdot \prod_{l\in L(G_t)} m(l)},$$

where d_G is the degree of the étale morphism $\overline{\mathscr{M}}_G \to \overline{\mathscr{M}}_{G_t}$. See Proposition 3.3.2.

Let f_k be the Cartier divisor on $\mathbb{R}^{\binom{n}{2}-n}$ related to ψ_k, i.e., $div(f_k) = \binom{n-1}{2}\psi_k$. Let

$$\tilde{f}_k : \mathscr{M}_\eta^{\text{trop}} \xrightarrow{\phi} \mathbb{R}^{\binom{n}{2}-n} \xrightarrow{f_k} \mathbb{R}$$

be the composition of ϕ and f_k.

Proposition 3.2.3. *On $\mathscr{M}_\eta^{trop}$, assume Y_h is the tropical fan associated to the Chow cohomology class $[h]$. Then $\tilde{f}_k \smile Y_h$ is the tropical fan associated to the Chow cohomology class*

$$m(k)\binom{n-1}{2}\psi_k \smile [h],$$

where $m(k)$ is the twisted degree of the marked leg $k \in L(\eta_t)$.

Proof. It suffices to prove the weight of $\tilde{f}_k \smile Y_h$ on $\mathscr{M}_G^{\text{trop}}$ is $m(k)\binom{n-1}{2}\psi_k \smile [h](\overline{\mathscr{M}}_G)$ for each type η graph cover G.

Let $\pi : \overline{\mathscr{M}}_\eta \to \overline{\mathscr{M}}_{0,n}$. The inverse image $\pi^{-1}(\overline{\mathscr{M}}_{r(G_t)})$ contains $\overline{\mathscr{M}}_G$ as a connected component. So the class $\pi^*\psi_k \smile \overline{\mathscr{M}}_G$ consists of classes lying in $\overline{\mathscr{M}}_G$.

Note that on $\overline{\mathscr{M}}_{0,n}$,

$$\binom{n-1}{2}\psi_k \smile \overline{\mathscr{M}}_{r(G_t)} = \sum_{H':\exists H' \twoheadrightarrow r(G_t)} \overline{\mathscr{M}}_{H'} f_{k,H'}(u_{H'/r(G_t)}) - \overline{\mathscr{M}}_{H'} f_{k,r(G_t)}(u_{H'/r(G_t)}),$$

where $f_{k,H'}$ means the linear extension of f_k restricted on H'. The intersection $\pi^*\psi_k \smile \overline{\mathscr{M}}_G$ consists of the class of $\pi^*(\psi_k \smile \overline{\mathscr{M}}_{c(G_t)})$ lying in $\overline{\mathscr{M}}_G$, i.e.,

$$\begin{aligned}&\binom{n-1}{2}\pi^*\psi_k \smile \overline{\mathscr{M}}_G\\ =&\sum_{H:\exists H\twoheadrightarrow G} \overline{\mathscr{M}}_H m(e_H)\tilde{f}_{k,H}(u_{H/G}) - \overline{\mathscr{M}}_H m(e_H)\tilde{f}_{k,G}(u_{H/G}),\end{aligned}$$

where $e_H \in E(H_t)$ is the unique edge contracted by $H \twoheadrightarrow G$. The $m(e_H)$ comes from the pullback formula

$$\pi^*\overline{\mathscr{M}}_K = \sum_{K':\exists r(K'_t)=K} m(e_{K'})\overline{\mathscr{M}}_{K'}$$

where K is a weighted marked graph and $e_{K'}$ is the unique edge of K'_t, which is a rephrase of [12, Lemma 4.0.2].

After plugging into $[h]$, the result is exactly the weight of $\tilde{f}_k \smile Y_h$. Moreover, the pullback $\pi^*\psi_k = m(k)\psi_k$. So $\binom{n-1}{2}\pi^*\psi_k \smile \overline{\mathscr{M}}_G = m(k)\binom{n-1}{2}\psi_k \smile \overline{\mathscr{M}}_G$ on $\overline{\mathscr{M}}_\eta$. □

In particular, the tropical fan Y_h is the fundamental tropical fan and $\tilde{f}_k$ gives the tropical fan associated to ψ_k when the cohomology class h is 1. From now on if there is no ambiguity, we will write f_k for $\tilde{f}_k$.

Definition 3.2.4. The psi-class ψ_k on $\mathscr{M}_\eta^{\mathrm{trop}}$ is the subcone stack consisting of all $(n-4)$-dimensional strata $\mathscr{M}_G^{\mathrm{trop}}$ whose k-marked leg in G_t has a root vertex of 4 valence. The weight on each stratum $\mathscr{M}_G^{\mathrm{trop}}$ is $\frac{d_G}{m(k)\prod_{e\in E(G_t)} m(e)\cdot\prod_{l\in L(G_t)} m(l)}$.

The pullback of ψ_k gives the part $\frac{1}{m(k)}$ where k is the unbounded edge marked by k while the rest of the weight comes from the pushforward in Proposition 3.3.2.

Let I be a type η graph cover where I_t has exactly two vertices. Then $\overline{\mathscr{M}}_I$ is an element in $A^1(\overline{\mathscr{M}}_\eta)$. We will give a Cartier divisor (piece-wise linear function) associated to it. Let v_I be the primitive vector in U_I, where $\mathscr{M}_I^{\mathrm{trop}} = (R_I \rightrightarrows U_I)$. Let f_I be the piecewise linear function on $\mathscr{M}_\eta$ such that

$$f_I(v_J) = \begin{cases} 1, & J = I\\ 0, & J \neq I\end{cases}$$

where J runs over all twisted covers of type η with two vertices. Let e_I be the unique edge in I_t connecting the two vertices.

Proposition 3.2.5. *Given $[h] \in A^*(\overline{\mathscr{M}}_\eta)$, the intersection $f_I \smile Y_h$ is the tropical fan associated to $m(e_I)[h] \smile \overline{\mathscr{M}}_I$, where $\overline{\mathscr{M}}_I$ is viewed as a chow cohomology class on $\overline{\mathscr{M}}_\eta$.*

Proof. It suffices to prove the weight of $f_I \smile Y_h$ on $\mathcal{M}_G^{\mathrm{trop}}$ is $\overline{\mathcal{M}}_I \smile [h](\overline{\mathcal{M}}_G)$.

If $\mathcal{M}_I^{\mathrm{trop}}$ is not a face of $\mathcal{M}_G^{\mathrm{trop}}$, then

$$\overline{\mathcal{M}}_I \smile \overline{\mathcal{M}}_G = \sum_{H:\exists H\twoheadrightarrow G, H\twoheadrightarrow I} \overline{\mathcal{M}}_H.$$

where H runs over all type η graph covers with $\dim \mathcal{M}_H^{\mathrm{trop}} = \dim \mathcal{M}_G^{\mathrm{trop}} + 1$. So

$$\overline{\mathcal{M}}_I \smile [h](\overline{\mathcal{M}}_G) = \sum_{H:\exists H\twoheadrightarrow G, H\twoheadrightarrow I} h(\overline{\mathcal{M}}_H).$$

On the other hand, the weight of $f_I \smile Y_h$ is

$$\sum_{H:\exists H\twoheadrightarrow G} h(\overline{\mathcal{M}}_H) m(e_I) f_{I,H}(u_{H/G}) - \sum_{H:\exists H\twoheadrightarrow G} f_{I,G}(m(e_I) h(\overline{\mathcal{M}}_H) u_{H/G}),$$

where $f_{I,H}$ is the linear extension of $f_I|_{\mathcal{M}_H^{\mathrm{trop}}}$. The part $\sum_{H:\exists H\twoheadrightarrow G} m(e_I) h(\overline{\mathcal{M}}_H) u_{H/G}$ lies in the space spanned by $\phi(\mathcal{M}_H^{\mathrm{trop}})$ by the balancing condition, hence has $f_{I,G}$ value 0. By definition, $f_{I,H}(u_{H/G}) = 1$. So

$$\sum_{H:\exists H\twoheadrightarrow G} h(\overline{\mathcal{M}}_H) m(e_I) f_{I,H}(u_{H/G}) = m(e_I) \overline{\mathcal{M}}_I \smile [h](\overline{\mathcal{M}}_G).$$

If $\mathcal{M}_I^{\mathrm{trop}}$ is a face of $\mathcal{M}_G^{\mathrm{trop}}$, then let g be the linear extension of $f_{I,G}$. Then g is associated to D_g, a pullback of a principal divisor on $\overline{\mathcal{M}}_{0,n}$. It suffices to prove the weight of $(f_I - g) \smile Y_h$ on $\mathcal{M}_G^{\mathrm{trop}}$ is $(\overline{\mathcal{M}}_I - D_g) \smile [h](\overline{\mathcal{M}}_G)$. Note that $f_I - g$ is a linear combination of f_J where $\mathcal{M}_J^{\mathrm{trop}}$ is not a face of $\mathcal{M}_G^{\mathrm{trop}}$. The proof is completed by the linearity of the $\smile$ operator on both sides. □

3.3 Application: Intersection Theory on $\overline{\mathcal{M}}_{g,n}$

In this section we use K.Costello's idea in [12] to do intersection theory on $\overline{\mathcal{M}}_{g,n}$ by applying tropical intersection theory. Although the cone stack $\mathcal{M}_{g,n}^{\mathrm{trop}}$ exists, it is difficult to define tropical intersection theory on $\mathcal{M}_{g,n}^{\mathrm{trop}}$ because $\overline{\mathcal{M}}_{g,n}$ does not embed nicely into a toric variety like $\overline{\mathcal{M}}_{0,n}$ or have a nice étale cover to some subvariety of a toric variety like $\overline{\mathcal{M}}_\eta$. So in order to perform intersection, the cycles are first pulled back to some $\overline{\mathcal{M}}_\eta$ and then intersected.

A special case of [12, Chapter 6] calculates the degree between the fundamental classes of $\overline{\mathscr{M}}_\eta$ and $\overline{\mathscr{M}}_v$. Let v be the type g,n. So $\overline{\mathscr{M}}_v$ is the moduli space of stable curves of genus g with n marked points. The type of covering $\eta = \eta_s \to \eta_t$ is defined by

- η_s, η_t have just one vertex,
- genus $g(\eta_s) = g$, $g(\eta_t) = 0$.
- The marking on η_t is

$$\{\infty\} \bigsqcup \{2,...,n\} \bigsqcup \{n+1,...,n+3g\}.$$

- The degree of η is $g+1$.
- The marking on η_s is

$$\{1\} \bigsqcup \{2,...,n\} \times [g+1] \bigsqcup (\{n+1,...,n+3g\} \times [g]),$$

  where for $N \in \mathbb{N}_{>0}$, $[N] := \{1,...,N\}$.
- The map $\eta_s \to \eta_t$ sends $1 \to \infty$, and is the natural projection on the other factors,

$$\{2,...,n\} \times [g+1] \to \{2,...,n\}, \{n+1,...,n+3g\} \times [g] \to \{n+1,...,n+3g\}.$$

- The multiplicity function on η_t is

$$m(i) = \begin{cases} g+1, & i = \infty \\ 1, & 2 \le i \le n \\ 2, & n+1 \le i \le n+3g. \end{cases}$$

- The multiplicity function on η_s is

$$m(i,j) = \begin{cases} 1, & (i,j) = 1 \\ 1, & 2 \le i \le n \\ 1, & n+1 \le i \le n+3g, 2 \le j \le g \\ 2, & n+1 \le i \le n+3g, j = 1. \end{cases}$$

The map $\overline{\mathscr{M}}_\eta \to \overline{\mathscr{M}}_v$ is taking the source curve and forgetting all other marking except 1 and $(i,1), i = 2,...,n$.

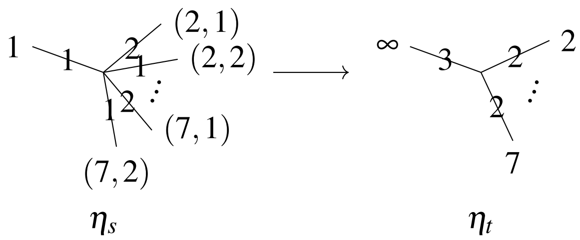
Example 3.3.1. Let v be the type $2,1$, i.e., genus 2 with 1 marked point. Then η_s and η_t have genus 2 and 0. The graph η_s has 13 legs labelled as

$$\{1,(2,1),(2,2),(3,1),(3,2),(4,1),(4,2),(5,1),(5,2),(6,1),(6,2),(7,1),(7,2)\}.$$

The graph η_t has 7 legs labelled as

$$\{\infty,2,3,4,5,6,7\}.$$

The mapping $\eta_s \to \eta_t$ maps 1 to ∞ and $(i,1),(i,2)$ to i for $i=2,\dots,7$. The multiplicity on η is as follows.



Proposition 3.3.2. *[12, Lemma 6.0.1] The map $\overline{\mathscr{M}}_\eta \to \overline{\mathscr{M}}_v$ is of degree*

$$\frac{(3g)!(g!)^{n-1}((g-1)!)^{3g}}{2^{3g}m(\infty)}.$$

The pullback of Psi-classes from $\overline{\mathscr{M}}_v$ to $\overline{\mathscr{M}}_\eta$ are calculated via another result from [12]. The morphism $\overline{\mathscr{M}}_\eta \to \overline{\mathscr{M}}_v$ decomposes as

$$\overline{\mathscr{M}}_\eta \xrightarrow{s} \overline{\mathscr{M}}_{\eta_s} \xrightarrow{r} \overline{\mathscr{M}}_{r(\eta_s)} \xrightarrow{\pi} \overline{\mathscr{M}}_v,$$

where the morphism s is taking the source curve, the morphism r is taking the coarse moduli curve and the morphism π is forgetting markings and stabilizing. Denote the set of forgotten markings by I. Assume γ is a weighted marked graph or twisted graph. For $e \in E(\gamma)$, let γ_e be the graph obtained by contracting all edges except e. The function $S(e,t,I) \in \{0,1\}$ is defined to be 1 if and only if t is in a vertex of γ_e that is contracted when stabilizing after forgetting the tails I [12, Lemma 4.1.1].

Proposition 3.3.3. *[12, Corollary 4.1.3] For each $l \in T(v)$,*

$$s^*r^*\pi^*\psi_l = m(l)\psi_l - \sum_{\gamma\to\eta}[\overline{\mathscr{M}}_\gamma]\sum_{e\in E(\gamma_s)} m(e)S(e,l,I)$$

where the sum is over all $\gamma \to \eta$ with $\#E(\gamma_t) = 1$.

The leg l on the right of the equation is the unique leg of η_s that corresponding to the marking l on v.

For each marking $k \in \{1,...,n\}$ of $\overline{\mathscr{M}}_{g,n}$, define the piecewise linear function on $\mathscr{M}_\eta^{\mathrm{trop}}$

$$g_k := \frac{m(k_s)}{m(k_t)\cdot \binom{3g+n-1}{2}} f_k - \sum_{\gamma:\exists\gamma\to\eta} \frac{1}{m(e_\gamma)} f_\gamma \sum_{e\in E(\gamma_s)} m(e)S(e,k_t,I),$$

where e_γ is the unique edge of γ_t. The $3g+n$ here is the number of marking on η_t. The leg k_s is the unique leg of η_s that corresponding to the marking k; the leg k_t is the image of k_s in η_t.

Theorem 3.3.4. *For $k_i \in \mathbb{N}$ satisfying $\sum_i k_i = 3g-3+n$, let X be obtained by intersecting the fundamental tropical fan of $\mathscr{M}_\eta^{trop}$ with each g_i for k_i times. Then the intersection number*

$$\int_{\overline{\mathscr{M}}_{g,n}} \prod_i \psi_i^{k_i}$$

is the same as the weight of X at the origin divided by the degree of $\overline{\mathscr{M}}_\eta \to \overline{\mathscr{M}}_{g,n}$ which is

$$\frac{(3g)!(g!)^{n-1}((g-1)!)^{3g}}{2^{3g}m(\infty)}.$$

Proof. Applying the projection formula and substituting the degree of $\overline{\mathscr{M}}_\eta \to \overline{\mathscr{M}}_{g,n}$ shows

$$\frac{(3g)!(g!)^{n-1}((g-1)!)^{3g}}{2^{3g}m(\infty)} \int_{\overline{\mathscr{M}}_{g,n}} \prod_i \psi_i^{k_i} = \int_{\overline{\mathscr{M}}_\eta} \prod_i s^*r^*\pi^*\psi_i^{k_i}.$$

By definition g_k is the piecewise linear function associated to $s^*r^*\pi^*\psi_i$. Given that $\sum_i k_i = 3g-3+n$, the tropical fan X is of dimension 0. So the weight of X at the origin is just the intersection number

$$\int_{\overline{\mathscr{M}}_\eta} \prod_i s^*r^*\pi^*\psi_i^{k_i}.$$

□

Example 3.3.5. We will show

$$\int_{\overline{\mathscr{M}}_{1,1}} \psi_1 = \frac{1}{24}.$$

We define the graph cover η as in [12, Example 9.2.1] as follows.

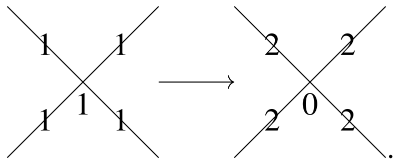
- The source graph η_s has 1 vertex v_s; the genus $g(\eta_t) = 0$ and $g(\eta_s) = 1$.

- The unbounded edges are

$$\begin{aligned} L(\eta_s) &= \{X_1, X_2, X_3, X_4\}, \quad d(v_s, X_i) = 2, \quad m(X_i) = 1; \\ L(\eta_t) &= \{x_1, x_2, x_3, x_4\}, \qquad m(x_i) = 2. \end{aligned}$$

- The map $L(\eta_s) \to L(\eta_t)$ sends $X_i \mapsto x_i$.

So the graph cover η looks like



Forgetting the marked points X_2, X_3, X_4 gives us the map

$$\pi \circ r \circ s : \overline{\mathscr{M}}_\eta \to \overline{\mathscr{M}}_{1,1}.$$

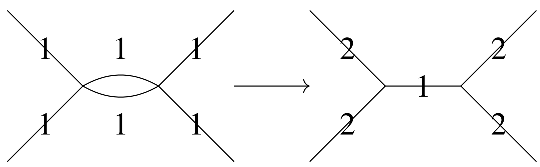
Apply Proposition 3.3.3 and notice that $S(e,t,I)$ is always 0 here. So

$$s^* r^* \pi^* \psi_1 = \psi_1.$$

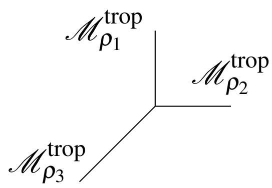
Apply Proposition 3.3.2 to get the degree of $\overline{\mathscr{M}}_\eta \to \overline{\mathscr{M}}_{1,1}$ is $\frac{3}{2^3}$. So by projection formula

$$\int_{\overline{\mathscr{M}}_{1,1}} \psi_1 = 3^{-1} \cdot 2^3 \int_{\overline{\mathscr{M}}_\eta} \psi_1.$$

The moduli space $\mathscr{M}_\eta^{\mathrm{trop}}$ consists of three 1-dimensional rays $\mathscr{M}_{\rho_i}, i = 1,2,3$ where the graph covers ρ_i are



with three different labeling of the legs. Moreover, the map $\mathscr{M}_\eta^{\text{trop}} \to \mathbb{R}^2$ induced from $\mathscr{M}_{0,4}^{\text{trop}} \to \mathbb{R}^2$ looks like



.

The piecewise linear function g_1 takes value $1/(m(1)\cdot\binom{3}{2}) = 1/6$ on the primitive vector of the three cones $\mathscr{M}_{\rho_i}^{\text{trop}}$. The fundamental tropical fan Δ associated to $\mathscr{M}_\eta^{\text{trop}}$ has weight

$$\omega(\rho_i) = \frac{d_{\rho_i}}{\prod_{e\in E(\rho_{it})} m(e)\cdot\prod_{l\in L(\rho_{it})} m(l)} = \frac{1/2}{2^4} = 2^{-5}.$$

The weight of the intersection $g_1 \smile \Delta$ is

$$\omega(\{0\}) = 2^{-5}\sum_{i=1}^{3} g_1(v_i) - g_1(\sum_{i=1}^{3} 2^{-5} v_i) = 2^{-5}\times(1/6)\times 3 - 0 = 2^{-6},$$

which means $\int_{\overline{\mathscr{M}}_\eta} \psi_1 = 2^{-6}$. So $\int_{\overline{\mathscr{M}}_{1,1}} \psi_1$ equals to $1/24$ by Theorem 3.3.4.

Example 3.3.6. We will show

$$\int_{\overline{\mathscr{M}}_{1,2}} \psi_1^2 = \int_{\overline{\mathscr{M}}_{1,2}} \psi_2^2 = \frac{1}{24},$$

$$\int_{\overline{\mathscr{M}}_{1,2}} \psi_1\psi_2 = \int_{\overline{\mathscr{M}}_{1,2}} \psi_2\psi_1 = \frac{1}{24}.$$

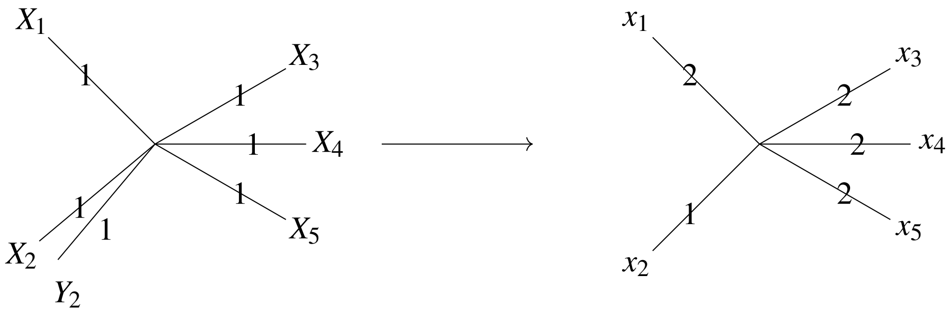
We define the graph cover η by

- η_s has 1 vertex; the genus $g(\eta_t) = 0$ and $g(\eta_s) = 1$.
- The unbounded edges and their multiplicities are

$$\begin{aligned}
&L(\eta_s) = \{X_1, X_2, Y_2, X_3, X_4, X_5\},\\
&m(X_i) = 1, i = 1,2,...,5, m(Y_2) = 1;\\
&L(\eta_t) = \{x_1, x_2, x_3, x_4, x_5\},\\
&m(x_2) = 1, m(x_i) = 2, i \neq 2.
\end{aligned}$$

- The map $L(\eta_s) \to L(\eta_t)$ sends $X_i \mapsto x_i$ and Y_2 to x_2.

So the graph cover η looks like



.

Let I be the set of forgotten markings $\{Y_2, X_3, X_4, X_5\}$. The forgetful map π forgets all the markings in I. Identifying the marking X_1 and X_2 with the marking 1 and 2, we have the map

$$\pi \circ r \circ s : \overline{\mathscr{M}}_\eta \to \overline{\mathscr{M}}_{1,2}.$$

To apply Proposition 3.3.3, we first need to calculate the piecewise linear function g_1 and g_2. For g_1, the coefficients of $[\overline{\mathscr{M}}_\gamma]$ are all 0 because $S(e, X_1, I)$ are all zero for any contraction $\gamma \to \eta$. We have

$$g_1 = \frac{1}{2 \times \binom{4}{2}} f_1 = \frac{1}{12} f_1.$$

For g_2, the coefficients of $[\overline{\mathscr{M}}_\gamma]$ can be nonzero. Indeed, for each $i = 3,4,5$ define graph cover ρ_i with a contraction $\rho_i \to \eta$.

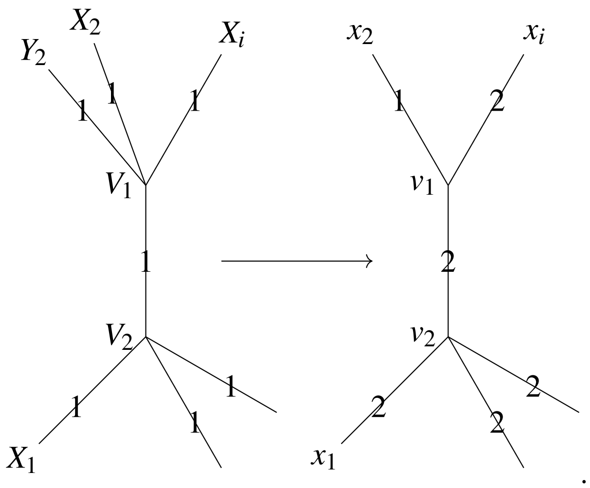
- The source ρ_{is} has 2 vertices V_1, V_2 with the genus $g(V_1) = 0$ and $g(V_2) = 1$. The target ρ_{it} has 2 vertices v_1, v_2. The map $\rho_{is} \to \rho_{it}$ sends $V_i \to v_i$.

- The unbounded edges and their multiplicities are the same as η. The set of unbounded edges of the vertices are given by

$$\begin{aligned} T(V_1) &= \{X_2, Y_2, X_i\}, & T(V_2) &= \{X_j | j \neq 2, i\}, \\ T(v_1) &= \{x_2, x_i\}, & T(v_2) &= \{x_i | i \neq 2, i\}. \end{aligned}$$

- There are two bounded edges E, e with $m(E) = 1$ and $m(e) = 2$. The edge E joins V_1 and V_2; the edge e joins v_1 and v_2.

So the graph cover ρ_i looks like



For each $\rho_i \to \eta$ contracting the edge e and E, the value of $S(E, X_2, I)$ is 1 because after forgetting the markings I, the marking X_2 is attached to V_1 which should be contracted during stabilization. We have

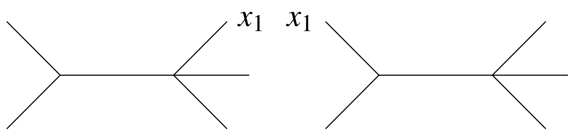
$$g_2 = \frac{1}{6} f_2 - \frac{1}{2} \sum_{i=3}^{5} f_{\rho_i}.$$

Note that the graph cover G of type η is completely determined by G_t. We will represent the graph cover G by its target G_t in the following discussion.

To give a explicit description of g_1 and g_2, it suffices to describe the value on the primitive generators of the 1-dimensional cones because g_1 and g_2 are piecewise linear. We denote the values by $g_1(G)$ and $g_2(G)$ for 1-dimensional cones $\mathscr{M}_G^{\mathrm{trop}}$. We have

$$g_1(G) = \begin{cases} \frac{1}{12}, & \text{if } x_1 \text{ is attached to a 4-valent vertex of } G_t \\ \frac{1}{4}, & \text{if } x_1 \text{ is not attached to a 4-valent vertex of } G_t \end{cases}$$

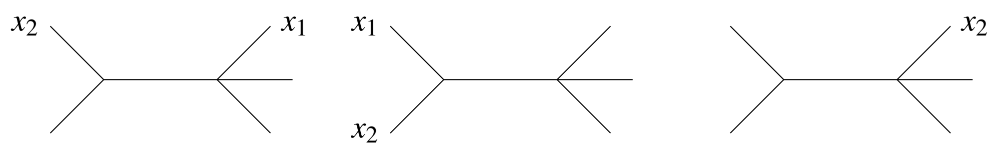
In the two cases, the targets G_t look like the following respectively.



For g_2, the situation is slightly complicated due to the existence of f_{ρ_i} terms. We have

$$g_2(G) = \begin{cases} 0, & \text{if } x_2 \text{ is attached to the 3-valent vertex while } x_1 \text{ is attached to the other vertex} \\ \frac{1}{2}, & \text{if } x_1 \text{ and } x_2 \text{ are both attached to the 3-valent vertex of } G_t \\ \frac{1}{6}, & \text{if } x_2 \text{ is attached to a 4-valent vertex.} \end{cases}$$

In the three cases, the targets G_t look like the following respectively.



Indeed, the value of $\frac{1}{6}f_2(G)$ is $\frac{1}{2}$ for the first two cases and $\frac{1}{6}$ for the last case while the value of $\frac{1}{2}\sum_{i=3}^{5} f_{\rho_i}$ is $\frac{1}{2}$ for the first case and zero for the rest two cases.

Now we are going to calculate the weights on the cones for the tropical fan $[\mathscr{M}_\eta^{\text{trop}}]$, $g_1 \smile [\mathscr{M}_\eta^{\text{trop}}]$, $g_2 \smile [\mathscr{M}_\eta^{\text{trop}}]$, $g_1 \smile g_1 \smile [\mathscr{M}_\eta^{\text{trop}}]$, $g_2 \smile g_1 \smile [\mathscr{M}_\eta^{\text{trop}}]$, $g_1 \smile g_2 \smile [\mathscr{M}_\eta^{\text{trop}}]$ and $g_2 \smile g_2 \smile [\mathscr{M}_\eta^{\text{trop}}]$.

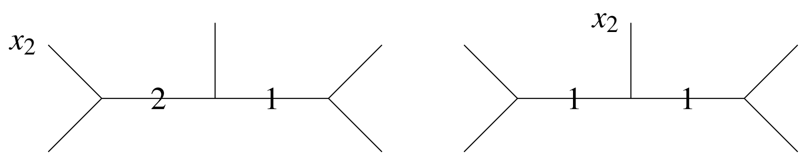
For fundamental tropical fan $[\mathscr{M}_\eta^{\text{trop}}]$, by definition, the weights are

$$\int_{\overline{\mathscr{M}}_G} [1] = \frac{d_G}{\prod_{e\in E(G_t)} m(e) \cdot \prod_{l\in L(G_t)} m(l)}$$

which are nonzero only for 2-dimensional cones $\mathscr{M}_G^{\text{trop}}$. We have $d_G = 1$ for all G. So we have

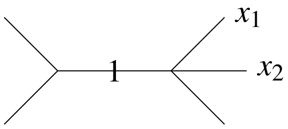
$$\omega_{[\mathscr{M}_\eta^{\text{trop}}]}(G) = \begin{cases} \frac{1}{2^5}, & \text{if } x_2 \text{ is not the only unbounded edge on the vertex} \\ \frac{1}{2^4}, & \text{if } x_2 \text{ is the only unbounded edge on the vertex} \end{cases}$$

In the two cases, the targets G_t look like the following respectively.



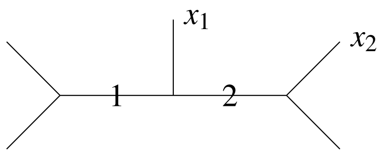
Note here if x_2 is not the only unbounded edge on the vertex it is attached to, then the multiplicities on the unbounded edges are forced to be 2 and 1 as pictured by the defining property of graph cover. The same applied to the case when x_2 is the only unbounded edge on the vertex. The extra $\frac{1}{2^4}$ comes from the multiplicities of the unbounded edges.

For $g_1 \smile [\mathscr{M}_\eta^{\mathrm{trop}}]$, the weights on different cones are calculated as the following. The target G_t is on the left while the weights are calculated on the right.

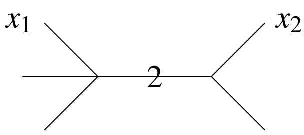


$$\begin{aligned} & 2\times\tfrac{1}{2^5}\times\tfrac{1}{12}+1\times\tfrac{1}{2^4}\times\tfrac{1}{4}+2\times\tfrac{1}{2^5}\times\tfrac{1}{4}-\tfrac{1}{2^4}\times\tfrac{1}{12} \\ = & \tfrac{1}{2^5}. \end{aligned}$$

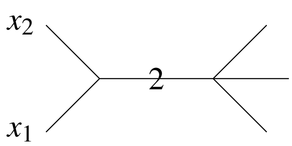
Here the multiplicity on the edge is forced to be 1 again by the defining properties of graph cover. The equation follows from the definition of the weight calculation. For example, in the first term $2\times\frac{1}{2^5}\times\frac{1}{12}$, the 2 is the multiplicity; the $\frac{1}{2^5}$ is the weight of



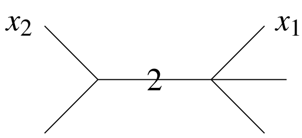
while the $\frac{1}{12}$ is the g_1 value on



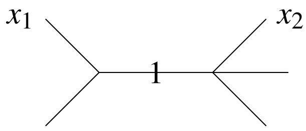
The last term $\frac{1}{2^4}\times\frac{1}{12}$ is the g_1 value for the weighted sum of the primitive vectors. The weights on the rest cones are:



$$\begin{aligned} & 1\times\tfrac{1}{2^5}\times\tfrac{1}{12}+1\times\tfrac{1}{2^5}\times\tfrac{1}{12}+1\times\tfrac{1}{2^5}\times\tfrac{1}{12}-\tfrac{1}{2^5}\times\tfrac{1}{4} \\ = & 0. \end{aligned}$$

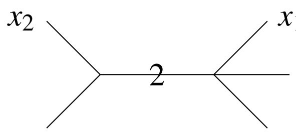


$$\begin{aligned} & 1\times\tfrac{1}{2^5}\times\tfrac{1}{12}+1\times\tfrac{1}{2^5}\times\tfrac{1}{4}+1\times\tfrac{1}{2^5}\times\tfrac{1}{4}-\tfrac{1}{2^5}\times\tfrac{1}{12} \\ = & \tfrac{1}{2^6}. \end{aligned}$$

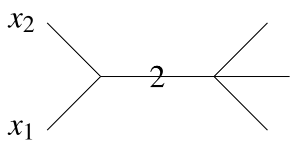


$$\begin{aligned} & 1\times\tfrac{1}{2^4}\times\tfrac{1}{12}+2\times\tfrac{1}{2^5}\times\tfrac{1}{12}+2\times\tfrac{1}{2^5}\times\tfrac{1}{12}-\tfrac{1}{2^4}\times\tfrac{1}{4} \\ = {} & 0. \end{aligned}$$

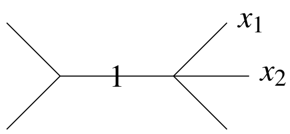
For $g_2 \smile [\mathscr{M}_\eta^{\mathrm{trop}}]$, the weights on the cones are:



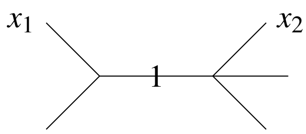
$$\begin{aligned} & 1\times\tfrac{1}{2^5}\times\tfrac{1}{6}+1\times\tfrac{1}{2^5}\times\tfrac{1}{6}+1\times\tfrac{1}{2^5}\times\tfrac{1}{6}-\tfrac{1}{2^5}\times 0 \\ = {} & \tfrac{1}{2^6}. \end{aligned}$$



$$\begin{aligned} & 1\times\tfrac{1}{2^5}\times\tfrac{1}{6}+1\times\tfrac{1}{2^5}\times\tfrac{1}{6}+1\times\tfrac{1}{2^5}\times\tfrac{1}{6}-\tfrac{1}{2^5}\times\tfrac{1}{2} \\ = {} & 0. \end{aligned}$$



$$\begin{aligned} & 1\times\tfrac{1}{2^4}\times\tfrac{1}{6}+2\times\tfrac{1}{2^5}\times 0+2\times\tfrac{1}{2^5}\times\tfrac{1}{2}-\tfrac{1}{2^4}\times\tfrac{1}{6} \\ = {} & \tfrac{1}{2^5}. \end{aligned}$$

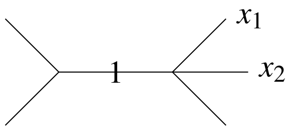


$$\begin{aligned} & 1\times\tfrac{1}{2^4}\times\tfrac{1}{6}+2\times\tfrac{1}{2^5}\times 0+2\times\tfrac{1}{2^5}\times 0-\tfrac{1}{2^4}\times\tfrac{1}{6} \\ = {} & 0. \end{aligned}$$

For $g_1 \smile g_1 \smile [\overline{\mathscr{M}}_\eta^{\mathrm{trop}}]$, the weight at the origin is

$$\begin{aligned} & 1\times\tfrac{1}{2^5}\times\tfrac{1}{12}\times 3+2\times 0\times\tfrac{1}{4}\times 1+2\times\tfrac{1}{2^6}\times\tfrac{1}{12}\times 3+1\times 0\times\tfrac{1}{4}\times 3 \\ = {} & \tfrac{1}{2^6}. \end{aligned}$$

The sum is over all 1-dimensional cones since they all contains the origin. Each term has the similar components. For example, in the first term $1 \times \frac{1}{2^5} \times \frac{1}{12} \times 3$ corresponding to



the 1 is the multiplicity of the unbounded edge; the $\frac{1}{2^5}$ is the weight; the $\frac{1}{12}$ is the g_1 value. The 3 is the number of different 1-dimensional cones that has this representation of G_t. Indeed, the last leg of the 4-valent vertex can be x_3, x_4 or x_5.

For $g_2 \smile g_1 \smile [\overline{\mathscr{M}}_{\eta}^{\text{trop}}]$, the weight at the origin is

$$\begin{aligned} & 1 \times \tfrac{1}{2^5} \times \tfrac{1}{6} \times 3 + 2 \times 0 \times \tfrac{1}{2} \times 1 + 2 \times \tfrac{1}{2^6} \times 0 \times 3 + 1 \times 0 \times \tfrac{1}{6} \times 3 \\ = \; & \tfrac{1}{2^6}. \end{aligned}$$

For $g_1 \smile g_2 \smile [\overline{\mathscr{M}}_{\eta}^{\text{trop}}]$, the weight at the origin is

$$\begin{aligned} & 2 \times \tfrac{1}{2^6} \times \tfrac{1}{12} \times 3 + 2 \times 0 \times \tfrac{1}{4} \times 1 + 1 \times \tfrac{1}{2^5} \times \tfrac{1}{12} \times 3 + 1 \times 0 \times \tfrac{1}{4} \times 3 \\ = \; & \tfrac{1}{2^6}. \end{aligned}$$

For $g_2 \smile g_2 \smile [\overline{\mathscr{M}}_{\eta}^{\text{trop}}]$, the weight at the origin is

$$\begin{aligned} & 2 \times \tfrac{1}{2^6} \times 0 \times 3 + 2 \times 0 \times \tfrac{1}{2} \times 1 + 1 \times \tfrac{1}{2^5} \times \tfrac{1}{6} \times 3 + 1 \times 0 \times \tfrac{1}{6} \times 3 \\ = \; & \tfrac{1}{2^6}. \end{aligned}$$

Finally, to get the intersection number on the schematic side, we apply Theorem 3.3.4 and times the result by

$$\begin{aligned} & \frac{(3g)!(g!)^{n-1}((g-1)!)^{3g}}{2^{3g} m(\infty)} \\ = \; & \frac{3!}{2^3 \times 2} = \frac{3}{8}. \end{aligned}$$

So the final result is $\frac{1}{24}$.

Chapter 4

Covers of Logarithmic Stable Curves

In this chapter we will discuss logarithmic stable curves and covers of logarithmic stable curves. First some basics about logarithmic stable curves and their moduli spaces are revisited. Then we review the tropicalization of the moduli space of logarithmic stable curves. In the end we discuss the covers of logarithmic stable curves, their moduli spaces and the tropicalization.

4.1 Moduli Space of Logarithmic Stable Curves

In this section we review the logarithmic stable curves and their moduli spaces.

Logarithmic geometry is first studied in [29] and [27].

Definition 4.1.1. A *monoid* is a commutative semi-group with a unit. A *morphism* of monoids $f : P \to Q$ is a map such that $f(0) = 0$ and $f(p+p') = f(p)+f(p')$.

Here 0 is the unit. The operator of the semi-group is '$+$'. Sometimes the unit is denoted by 1 while the operator is denoted by '$\cdot$'.

The *Grothendieck group* of a monoid P is

$$P^{gp} := \{p-p' \mid p, p' \in P\}/\sim,$$

where $p-p'$ is a formal symbol. The equivalence is that $p-p' \sim q-q'$ if and only if there exists $r \in P$ such that $p+q'+r = q+p'+r$.

A monoid P is *integral* if $P \to P^{gp}$ is injective.

A monoid is *fine* if it is finitely generated and integral. A monoid P is *saturated* if it is integral and

$$mp \in P \Rightarrow p \in P$$

for any $p \in P^{gp}$ and $m \in \mathbb{N}_+$. See [21, Subsection 3.1.1] for more details.

Definition 4.1.2. A *logarithmic structure* on a scheme X is a sheaf of monoids M_X on X together with a morphism of sheaves of monoids

$$\alpha_X : M_X \to \mathscr{O}_X$$

with respect to the multiplication on $\mathscr{O}_X$ such that

$$\alpha_X : \alpha_X^{-1}(\mathscr{O}_X^\times) \to \mathscr{O}_X^\times$$

is an isomorphism. Here $\mathscr{O}_X^\times$ denotes the sheaf of invertible elements of $\mathscr{O}_X$.

For any morphism of sheaf of monoids $\alpha_X : M_X \to \mathscr{O}_X$, we can construct a logarithmic structure associated to it. Given a morphism of schemes $f : \underline{X} \to \underline{Y}$ and the logarithmic structure M_Y, the *pullback logarithmic structure*

$$\alpha_X : f^* M_Y \to \mathscr{O}_X$$

on $\underline{X}$ is the logarithmic structure associated to the composition $f^{-1}M_Y \to f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. For more detail, see [27, chapter 1].

For a logarithmic scheme X, we denote the underlying scheme by $\underline{X}$.

Given an underlying scheme $\underline{X}$, there are many different possible log structures.

Example 4.1.3. Let $D \subseteq \underline{X}$ be a closed subset of pure codimension one. Let $j : \underline{X} \setminus D \hookrightarrow X$ be the inclusion, and set

$$M_{(\underline{X},D)} := (j_* \mathscr{O}_{\underline{X}\setminus D}^\times) \cap \mathscr{O}_{\underline{X}}.$$

This is the sheaf of regular functions on $\underline{X}$ which are invertible on $\underline{X} \setminus D$. Define $\alpha_X : M_{(\underline{X},D)} \hookrightarrow \mathscr{O}_{\underline{X}}$ as the inclusion. This is called the *divisorial logarithmic structure* associated to $(\underline{X}, D)$.

Example 4.1.4. [21, Example 3.19] The *toric log structure* on a toric variety $\underline{X}$ is the divisorial log structure associated to $(\underline{X_\Sigma}, \underline{X_\Sigma} \backslash T)$ where T is the big torus of the toric variety. Assume $\underline{X}$ is the toric variety associated to the fan Σ. For a cone σ of Σ, the toric log structure on the open set $\mathrm{Spec}\mathbb{k}[\sigma^\vee \cap M]$ is induced by the chart $\underline{\sigma^\vee \cap M} \to \mathscr{O}_{\mathrm{Spec}\mathbb{k}[\sigma^\vee \cap M]}$.

A *morphism* of logarithmic schemes $f : X \to Y$ is a morphism of schemes $\underline{f} : \underline{X} \to \underline{Y}$ along with a morphism of sheaves of monoids $f^\# : \underline{f}^{-1} M_Y \to M_X$ such that the following diagram commutes.

$$\begin{tikzcd}
\underline{f}^{-1}M_Y \arrow[r, "f^{\#}"] \arrow[d, "\alpha_Y"] & M_X \arrow[d, "\alpha_X"] \\
\underline{f}^{-1}\mathscr{O}_Y \arrow[r, "\underline{f}^{*}"] & \mathscr{O}_X
\end{tikzcd}$$

Here $\underline{f}^*$ is the usual pullback of regular functions defined by the morphism $\underline{f}$.

A logarithmic morphism $f : X \to Y$ is *strict* if $f^{\#} : f^{-1}M_Y \to M_X$ induces an isomorphism between the pull-back of the logarithmic structure on Y to X and the logarithmic structure on X.

For simplicity, we will use "log" instead of "logarithmic".

Definition 4.1.5. The *ghost sheaf* $\overline{M}_X$ of X is the sheaf of monoids given by the exact sequence

$$1 \longrightarrow \mathscr{O}_X^{\times} \xrightarrow{\alpha_X^{-1}} M_X \longrightarrow \overline{M}_X \longrightarrow 0.$$

The following proposition is useful in studying the pullback of log structures.

Proposition 4.1.6. *Let $f : \underline{X} \to \underline{Y}$ be a morphism of schemes. The log structure on X is defined by $M_X := f^*M_Y$. Then*

$$\overline{M}_X = f^{-1}\overline{M}_Y.$$

Proof. See [21, Section 3.2]. $\square$

Definition 4.1.7. Let X be a scheme. Let P be a monoid and denote by $\underline{P}$ the constant sheaf on X with stalk P. Given a pre-log structure $\pi : \underline{P} \to \mathscr{O}_X$, the map π is called *the chart* of its associated log structure.

We also say $\pi : \underline{P} \to \mathscr{O}_X$ is a chart for M_X if the associated log structure of π is isomorphic to M_X.

We will focus on log structures with certain nice properties.

A log structure M_X is *fine* if étale locally M_X has charts from fine monoids. Here "étale locally" simply means there is an étale open cover $\{U_i\}$ of $\underline{X}$ such that the pullback of M_X to each U_i has the property.

A log structure M_X is saturated if $\overline{M}_{X,x}$ is saturated for all closed point $x \in X$.

Definition 4.1.8. [21, Definition 3.23] A morphism $f : X \to Y$ of fine log schemes is *log smooth (étale)* if étale locally on $\underline{X}$ and $\underline{Y}$ there is a commutative diagram

$$\begin{tikzcd} \underline{X} \arrow[r] \arrow[d] & \mathrm{Spec}\mathbb{Z}[P] \arrow[d] \\ \underline{Y} \arrow[r] & \mathrm{Spec}\mathbb{Z}[Q] \end{tikzcd}$$

with the following properties:

(1) The horizontal maps induce charts $\underline{P} \to \mathscr{O}_X$ and $\underline{Q} \to \mathscr{O}_Y$ for X and Y.

(2) The induced morphism

$$\underline{X} \to \underline{Y} \times_{\mathrm{Spec}\mathbb{Z}[Q]} \mathrm{Spec}\mathbb{Z}[P]$$

is a smooth (étale) morphism of schemes.

(3) The right-hand vertical arrow is induced by a monoid homomorphism $Q \to P$ with $\ker(Q^{gp} \to P^{gp})$ and the torsion part of $\mathrm{coker}(Q^{gp} \to P^{gp})$ finite groups of orders invertible on X.

This definition is equivalent to the original definition from [29, Section 3.3] by [29, Theorem 3.5].

If we equip $\underline{X}$ and $\underline{Y}$ with *the trivial log structure*, i.e.,

$$\alpha_X : \mathscr{O}_X^* \to \mathscr{O}_X, \alpha_Y : \mathscr{O}_Y^* \to \mathscr{O}_Y,$$

then the morphism of log schemes $X \to Y$ is log smooth (étale) if and only if the underlying morphism of schemes $X \to Y$ is smooth (étale) [29, Proposition 3.8].

Some log morphisms are log smooth even if the underlying scheme morphism is not smooth.

Example 4.1.9. Let log scheme Y be $\mathrm{Spec}\Bbbk$ with the trivial log structure, which is induced by chart $Q = 0$. Let $\underline{X} = \mathrm{Spec}\Bbbk[P]$ with P a toric monoid, i.e., $P = \sigma^\vee \cap M$ for some cone (N, σ). Equip $\underline{X}$ with the log structure induced by $\underline{P} \to \mathscr{O}_{\mathrm{Spec}\Bbbk[P]}$. Then the morphism $\underline{X} \to \underline{Y} \times_{\mathrm{Spec}\Bbbk[Q]} \mathrm{Spec}\Bbbk[P]$ is an isomorphism, hence smooth (étale). So X is always log smooth (étale) over Y. Of course, the toric variety $\underline{X}$ is not necessarily étale or even smooth over $\mathrm{Spec}\Bbbk$.

Example 4.1.10. Let Y be *the standard log point*, i.e., the scheme $\mathrm{Spec}\Bbbk$ with the log structure induced by

$$\underline{\mathbb{N}} \to \mathscr{O}_{\mathrm{Spec}\Bbbk}$$

where $1 \mapsto 0$. Let $P = \mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2$ be given by $1 \mapsto 1$ and $1 \mapsto (1,1)$. Let the log scheme X be $\mathrm{Spec}\Bbbk[x,y]/(xy)$ equipped with log structure induced by

$$\underline{P} \to \mathscr{O}_X$$

where $(a,(b,c)) \mapsto x^{a+b}y^{a+c}$. The log morphism $X \to Y$ is given by $\mathbb{N} \to P$ with $1 \mapsto (1,(0,0))$. Then this morphism is log smooth (étale). Indeed, the product is

$$\Bbbk[P] \times_{\Bbbk[\mathbb{N}]} \Bbbk = k[x,y]/(xy).$$

So the morphism $\underline{X} \to \underline{Y} \times_{\mathrm{Spec}\Bbbk[\mathbb{N}]} \mathrm{Spec}\Bbbk[P]$ is an isomorphism.

A *log curve* with base S is a log smooth and integral morphism $f: X \to S$ of fine saturated log schemes such that every geometric fiber of the underlying morphism $\underline{f}$ is a reduced and connected curve. This is exactly [28, Definition 1.2]. For details about integral morphisms, see [29, Proposition 4.1].

There is characterization for the local structure of log smooth curves.

Theorem 4.1.11. *[28, Section 1.8] Let $f: X \to S$ be a log smooth curve with $S = \mathrm{Spec}\Bbbk$. Denote the only point in $\mathrm{Spec}\Bbbk$ by s. Assume $\overline{M}_{S,s} = Q$. The log structure on S must be induced by the chart $\sigma : \underline{Q} \to \mathscr{O}_S$,*

$$\sigma(q) = \begin{cases} 1, \textit{if } q \textit{ is the unit} \\ 0, \textit{otherwise} \end{cases}$$

Then X is étale locally isomorphic to one of the following log schemes:

(1) $V = \mathrm{Spec}\Bbbk[u]$, where the log structure is induced by the chart

$$Q \to \mathscr{O}_V, q \mapsto f^*\sigma(q).$$

(2) $V = \mathrm{Spec}\Bbbk[u]$, where the log structure is induced by the chart

$$Q \oplus \mathbb{N} \to \mathscr{O}_V, (q,a) \mapsto u^a f^*\sigma(q).$$

(3) $V = \mathrm{Spec}\Bbbk[u,v]/(uv)$, where the log structure is induced by the chart

$$Q \oplus_{\mathbb{N}} \mathbb{N}^2 \to \mathscr{O}_V, (q,(a,b)) \mapsto u^a v^b f^* \sigma(q).$$

Here $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ is defined using map $\mathbb{N} \to \mathbb{N}^2, 1 \mapsto (1,1)$ and $\mathbb{N} \to Q, 1 \mapsto \alpha \in Q$ with $\sigma(\alpha) = 0$.

Theorem 4.1.11 gives a local description of log smooth curves. The points of types (1), (2) and (3) on X are called *smooth points*, *log marked points* and *double points* respectively.

Log smooth curves are closely related to stable curves.

Definition 4.1.12. [23, Definition 1.3] A *pre-stable (marked) log curve* over W is a pair $(C/W, \mathbf{x})$ consisting of a proper log smooth and integral morphism $\pi : C \to W$ of fine saturated log schemes over S together with a tuple of sections $\mathbf{x} = (x_1, x_2, \ldots, x_l)$ of $\underline{\pi}$, such that every geometric fibre of π is a reduced and connected curve, and if $U \subset \underline{C}$ is the non-critical locus of π, then $\overline{M}_C|_U \cong \underline{\pi}^* \overline{M}_W \oplus_i x_{i*} \mathbb{N}_{\underline{W}}$.

A pre-stable log curve is *stable* if forgetting the log structure leads to an ordinary stable curve.

Given a stable curve, there are too many possible log structures making the stable curve log smooth, i.e., the *stack of stable log curves* $\tilde{\mathscr{M}}_{g,n}^{\log}$ is usually not Deligne-Mumford.

In order to have a nice moduli space, we need the basicness condition. This is [28, Proposition 2.3]. For a more general basicness condition of log stable maps, see [23, Section 1.5].

Definition 4.1.13. Recall the local structure for log smooth curves. For $\underline{W} = \mathrm{Spec}\Bbbk$, a stable curve $(\pi : C \to W, \mathbf{x})$ is *basic* if

- the chart is given by
$$Q = \prod_q \mathbb{N}$$
where q runs over all nodes;
- marked points are precisely log marked points;
- nodes are precisely double points with $\mathbb{N} \to Q$ being $1 \mapsto (0, \ldots 1, \ldots, 0)$ mapping to the corresponding entry at each node q;
- all other points are smooth points.

In general, $(\pi : C \to W, \mathbf{x})$ is *basic* if for each geometric point $\underline{w} \to \underline{W}$ equipped with the standard log structure, the log stable curve $w \times_W C \to w$ is basic.

Proposition 4.1.14. *The stack $\mathscr{M}^{log}_{g,n}$ of basic log stable curves is Deligne-Mumford.*

Proof. This is a special case of [23, Corollary 2.8] if X is set to be $\mathrm{Spec}\mathbb{k}$ with trivial log structure. $\square$

Denote the category of schemes, log schemes and fine saturated log schemes by **Sch**, **LSch** and $\mathbf{LSch}^{\mathrm{fs}}$ respectively.

Definition 4.1.15. [28, Definition 3.1]

(1) A log structure on the stack $\mathscr{S} \to \mathbf{Sch}$ is a covariant functor $L : \mathscr{S} \to \mathbf{LSch}$ which makes the triangle

$$\begin{tikzcd} \mathscr{S} \arrow[r] \arrow[dr] & \mathbf{LSch} \arrow[d] \\ & \mathbf{Sch} \end{tikzcd}$$

commute such that for any morphism $f : x \to y$ in $\mathscr{S}$, the morphism $L(f) : L(x) \to L(y)$ in **LSch** is strict. Here $\mathbf{LSch} \to \mathbf{Sch}$ is the forgetful functor.

(2) A log stack $(\mathscr{S}, L)$ is a log stack $\mathscr{S} \to \mathbf{Sch}$ with a log structure L on $\mathscr{S}$.

We will denote the log stack $(\mathscr{S}, L)$ by $\mathscr{S}^\dagger$. If **LSch** is replaced by $\mathbf{LSch}^{\mathrm{fs}}$, we say L is a fine saturated log structure and $\mathscr{S}^\dagger$ a fine saturated log stack.

There is a description of $\mathscr{M}^{\log}_{g,n}$ in terms of the log stack.

Theorem 4.1.16. *[28, Theorem 4.5] The stack $\mathscr{M}^{log}_{g,n}$ is naturally represented by the log stack $(\overline{\mathscr{M}}_{g,n}, L)$ with the fine saturated log structure L equipping stable curve $(\underline{C} \to \underline{W}, \mathbf{x})$ with the basic log structure.*

4.2 Logarithmic Étale Covers of Logarithmic stable Curves

In this section we will investigate log étale covers of log stable curves. In particular, we will define the basic log étale covers. Moreover, we will show the log structure on any log étale cover arises as the pull-back from a basic étale cover with the same underlying morphism following the idea in [23, Section 1.5]. In the end, we will define the moduli stack of log étale covers as the log stack $(\mathscr{M}^{\log}_\eta, L_\eta)$ which is the moduli stack $\mathscr{M}^{\log}_\eta$ of basic log étale covers equipped with the natural log structure L_η.

Definition 4.2.1. A log étale cover $(C_s \to C_t \to W, \mathbf{x}_s, \mathbf{x}_t)$ of log stable curves is a log étale cover $C_s \to C_t$ of log stable curves over W such that the inverse of the marked points $\mathbf{x}_t$ is precisely the set of marked points $\mathbf{x}_s$ while at the double points the underlying log morphism is given by

$$\mathrm{Spec}\mathbb{k}[u_s, v_s]/(u_s v_s) \to \mathrm{Spec}\mathbb{k}[u_t, v_t]/(u_t v_t), u_s \mapsto u_s^d, v_s \mapsto v_s^d$$

for some $d \in \mathbb{N}_+$.

We write $C_s \to C_t$ for a log étale cover when there is no ambiguity.

Proposition 4.2.2. *Recall Theorem 4.1.11. Given a log étale cover $C_s \to C_t$, then étale locally the morphism is one of the following:*

(1) The underlying morphism is

$$Spec\mathbb{k}[u_s] \to Spec\mathbb{k}[u_t], u_t \mapsto u_s.$$

The log morphism is induced by the identity $Q \to Q$.

(2) The underlying morphism is

$$Spec\mathbb{k}[u_s] \to Spec\mathbb{k}[u_t], u_t \mapsto u_s^d,$$

where $d \in \mathbb{N}_+$. The log morphism is induced by

$$Q \oplus \mathbb{N} \to Q \oplus \mathbb{N}, (q, a) \mapsto (q, da).$$

(3) The underlying morphism is

$$Spec\mathbb{k}[u_s, v_s]/(u_s v_s) \to Spec\mathbb{k}[u_t, v_t]/(u_t v_t), u_t \mapsto u_s^d, v_t \mapsto v_s^d,$$

where $d \in \mathbb{N}_+$. The log morphism is induced by

$$Q \oplus_{\mathbb{N}} \mathbb{N}^2 \to Q \oplus_{\mathbb{N}} \mathbb{N}^2, (q, (a, b)) \mapsto (q, (da, db)).$$

Here $d\alpha_s = \alpha_t$.

Proof. Because $C_s \to C_t$ is log étale, the double points must map to double points. By definition, the marked points map to marked points. So the smooth points map to smooth points.

Recall Q induces the log structure on the base W. So the log morphism $C_s \to C_t$ always sends Q to Q via identity.

For marked points, étale locally the underlying morphism is

$$\mathrm{Spec}\,\mathbb{k}[u_s] \to \mathrm{Spec}\,\mathbb{k}[u_t], u_t \mapsto u_s^d$$

for some $d \in \mathbb{N}_+$. The chart morphism then has to send $1 \mapsto d$ for the $\mathbb{N}$ factor by commutativity with the underlying scheme morphism.

For double points, the underlying morphism is specified in Definition 4.2.1. The chart morphism then has to send $(a,b) \mapsto (da, db)$ by commutativity with the underlying scheme morphism. The extra condition then follows from the identity map on Q by

$$(\alpha_t, (0,0)) = (0, (1,1)) \mapsto (0, (d,d)) = (d\alpha_s, (0,0)).$$

$\square$

Given a marked or double point p on C_t, denote those d appearing in the Proposition 4.2.2 by d_q for each q lying in the inverse image of p. Set m_p to be the least common multiple of d_q and set

$$m_q = m_p/d_q.$$

The dual intersection graph of a log étale cover is the graph cover with the underlying morphism of weighted marked graph being the dual intersection graph of the curve morphism $\underline{C_s} \to \underline{C_t}$ and the twisted degree at p, q being m_p, m_q respectively.

Given a graph cover η whose η_s has 1 vertex and no edges, a log étale cover is *of type* η if its dual intersection graph has a contraction to η.

Given a log étale cover $\pi : C_s \to C_t$ over a point, the log structure is *basic* if

- on the base it is induced by
$$Q^{basic} = \prod_p \mathbb{N}$$
where p runs over all nodes in $\underline{C_t}$;
- at a node p on C_t, the log structure is induced by $Q^{basic} \oplus_{\mathbb{N}} \mathbb{N}^2$ where $\mathbb{N} \to Q^{basic}$ is determined by $1 \mapsto \alpha_p = (0, ..., m_p, ..., 0) \in Q^{basic}$;
- at a node q on C_s, the log structure is induced by $Q^{basic} \oplus_{\mathbb{N}} \mathbb{N}^2$ where $\mathbb{N} \to Q^{basic}$ is determined by $1 \mapsto \alpha_q = (0, ..., m_q, ..., 0) \in Q^{basic}$ with the nonzero entry where m_q sits being the entry corresponding to $\pi(q)$.

The monoid Q^{basic} is called the *basic monoid* of the underlying morphism of stable curves.

Proposition 4.2.3. *The basic monoid Q^{basic} of $\pi : \underline{C_s} \to \underline{C_t}$ is the initial object of all monoids Q which make $C_s \to C_t \to (Speck, Q)$ log étale covers.*

Proof. Let $\{q_i\} \subset \underline{C_s}$ be all the nodes lying over $p \in \underline{C_t}$. If monoid Q makes $C_s \to C_t \to (\mathrm{Spec}k, Q)$ a log étale cover, then we have

$$d_{q_i}\alpha_{q_i} = \alpha_p \in Q.$$

Let m_p be the least common multiple of d_{q_i}. By saturatedness of Q, we can find unique α'_p such that $\alpha_p = m_p\alpha'_p$. Define

$$Q^{basic} \to Q$$

to be $1 \mapsto \alpha'_p$ at the entry corresponding to p. This is canonical and commutes with morphisms between Q. □

Given a log étale cover $(C_s \to C_t \to W)$ and a geometric point $\underline{w} \to \underline{W}$, the *pullback log étale cover*

$$\underline{w} \times_{\underline{W}} (C_s \to C_t \to W)$$

is $(\underline{w} \times_{\underline{W}} \underline{C_s} \to \underline{w} \times_{\underline{W}} \underline{C_t} \to \underline{w})$ equipped with the pullback log structure.

Definition 4.2.4. A log étale cover $(C_s \to C_t \to W)$ is called *basic* if for any geometric point $\underline{w} \to \underline{W}$, the log structure of the pullback étale cover $\underline{w} \times_{\underline{W}} (C_s \to C_t \to W)$ is basic.

We will show basicness is an open condition.

Proposition 4.2.5. *Given a log étale cover $(C_s \to C_t \to W)$, the set*

$$\Omega := \{\underline{w} \in \underline{W} | Spec\kappa(\underline{w}) \times_{\underline{W}} (C_s \to C_t \to W) \text{ is basic}\}$$

is an open subset of $\underline{W}$.

Proof. The set Ω is constructible because basicness is a condition on morphisms of fine sheaves. It suffices to show Ω is closed under generization, i.e., if $\underline{\omega_1} \in \Omega$, $\underline{\omega_2} \in \underline{W}$ and $\underline{\omega_1} \in \mathrm{cl}(\underline{\omega_2})$, then $\underline{\omega_2} \in \Omega$.

We first reduce it to a simple case. Since basicness is stable under strict base change, we may first replace $\underline{W}$ by $\mathrm{Spec}(\mathscr{O}_{\underline{W},\underline{\omega_1}})$ and then by $\mathrm{cl}(\underline{\omega_2})$ with the induced reduced scheme structure, to reduce to the case $\underline{W} = \mathrm{Spec}R$ for a strictly Henselian local domain R, and with $\underline{\omega_1}$ and $\underline{\omega_2}$ the closed point 0 and $\mathrm{Spec}K$ where K is the quotient field of R. Denote by

$\kappa = R/\mathfrak{m}$ the residue field of R and endow $\mathrm{Spec}\kappa$ and $\mathrm{Spec}K$ with the pullback log structure from W.

By assumption the log structure on $0 \times_{\underline{W}} (C_s \to C_t \to W)$ is basic and $\mathrm{Spec}K \times_{\underline{W}} (C_s \to C_t \to W)$ is a log étale cover. Assume the log structures on the bases are given by Q_0 and Q_K. Then we have a commutative diagram

$$\begin{array}{ccc} \prod_q \mathbb{N} & \xrightarrow{\theta} & Q_0 \\ \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \phi} \\ \prod_{\hat{q}} \mathbb{N} & \xrightarrow{\hat{\theta}} & Q_K \end{array}$$

where $q, \hat{q}$ runs over all nodes of $0 \times_{\underline{W}} \underline{C_t}$, $\mathrm{Spec}K \times_{\underline{W}} \underline{C_t}$ respetively, the morphism ϕ is the generization, the morphism θ is the identity by basicness, the morphism $\hat{\theta}$ is the canonical morphism from Proposition 4.2.3 and the morphism $\tilde{\phi}$ maps $1 \mapsto 1$ at the entries $q \in \mathrm{cl}(\hat{q})$ and $1 \mapsto 0$ at the entries otherwise.

Notice if $q \in \mathrm{cl}(\hat{q})$, then $\alpha_q \mapsto \alpha_{\hat{q}}$ under $\tilde{\phi}$. So it suffices to show $\hat{\theta}$ is an isomorphism. Indeed, the morphism ϕ induces a isomorphism

$$S^{-1}Q/(S^{-1}Q)^{\times} \xrightarrow{\simeq} Q_K$$

where $S = \phi^{-1}(0)$. All elements in $\prod_{\hat{q}} \mathbb{N}$ except 0 maps to uninvertible elements in Q_K under $\hat{\theta}$. So S is equal to $\tilde{\phi}^{-1}(0)$ as well. Hence $\hat{\theta}$ is an isomorphism. The local description of log étale cover then shows $\mathrm{Spec}K \times_{\underline{W}} (C_s \to C_t \to W)$ is a basic log étale cover. □

Notice that the proof actually shows fiberwise defined Q_K are compatible with generization. We are now in position to proof the universal property for basic log étale covers.

Proposition 4.2.6. *Any log étale cover arises as the pull-back from a basic log étale cover with the same underlying morphism of stable curves. Both the basic log étale cover and the morphism are unique up to unique isomorphism.*

Proof. Assume the log étale cover is $C_s \to C_t \to W$. We first define the Q^{basic} fiberwise over $\underline{W}$. Notice this definition is compatible with generization. So we can define the basic log étale cover for $C_s \to C_t \to W$ fiberwisely according to the local chart, e.g., if the local chart for $C_s \to C_t$ is

$$\begin{array}{ccc} Q \oplus_{\mathbb{N}} \mathbb{N}^2 & \longrightarrow & A[x,y]/(xy-t) \\ \downarrow & & \downarrow \\ Q \oplus_{\mathbb{N}} \mathbb{N}^2 & \longrightarrow & A[x',y']/(x'y'-t') \end{array}$$

where A is a strictly Henselian ring and t lies in the maximal ideal of A, then define the basic log structure by replacing Q with Q^{basic}. We hence define the basic log structure. Moreover, we have a morphism on the level of ghost sheaf from the basic log structure to the original log structure.

If $M_1 \to M_2$ is a morphism of fine log structures on a scheme Y, then from the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathscr{O}_Y^\times \arrow[r] \arrow[d, equal] & M_1^{gp} \arrow[r] \arrow[d] & \overline{M}_1^{gp} \arrow[r] \arrow[d] & 0 \\ 1 \arrow[r] & \mathscr{O}_Y^\times \arrow[r] & M_2^{gp} \arrow[r] & \overline{M}_2^{gp} \arrow[r] & 0 \end{tikzcd}$$

it follows that $M_1 = M_2 \times_{\overline{M}_2} \overline{M}_1$. Moreover, a morphism $M \to M_2$ of log structures lifts to $M \to M_1 \to M_2$ if and only if $\overline{M} \to \overline{M}_2$ lifts to $\overline{M} \to \overline{M}_1 \to \overline{M}_2$. So we have a morphism unique up to unique isomorphism on the level of log structure. □

Definition 4.2.7. The stack of type η log étale covers is the category $\mathscr{M}_\eta^{\log}$ of type η basic log étale covers together with the forgetful morphism $\mathscr{M}_\eta^{\log} \to \mathbf{Sch}$ mapping $(C_s \to C_t \to W, \mathbf{x}_s, \mathbf{x_t})$ to $\underline{W}$.

There is a natural fine saturated log structure $L_\eta : \mathscr{M}_\eta^{\log} \to \mathbf{LSch}^{\mathrm{fs}}$ mapping $(C_s \to C_t \to W, \mathbf{x}_s, \mathbf{x_t})$ to W.

Recall $\overline{\mathscr{M}}_{r(\eta_t)}$ is actually $\overline{\mathscr{M}}_{g,n}$ for some g, n. Let L be the log structure on $\overline{\mathscr{M}}_{r(\eta_t)}$ defined in Theorem 4.1.16.

Proposition 4.2.8. *The natural log morphism $(\mathscr{M}_\eta^{log}, L_\eta) \to (\overline{\mathscr{M}}_{r(\eta_t)}, L)$ is log étale .*

Proof. It suffices to verify the lifting property [29, Chapter 3]. To be more precise, it suffices to prove that for every diagram

$$\begin{tikzcd} T' \arrow[r] \arrow[d] & (\mathscr{M}_\eta^{\log}, L_\eta) \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & (\overline{\mathscr{M}}_{r(\eta_t)}, L) \end{tikzcd}$$

of solid arrows there is a unique lift of the dotted arrow where $T' \to T$ is an exact closed immersion (i.e., the underlying scheme morphism $\underline{T}' \to \underline{T}$ is a closed immersion and the log morphism $T' \to T$ is strict) with $\underline{T}'$ defined in $\underline{T}$ by an ideal I such that $I^2 = (0)$.

Giving a morphism $T \to (\mathscr{M}_\eta^{\log}, L_\eta)$ is the same as giving a log étale cover $C_s \to C_t \to T$. So to prove the lifting property, we need to show there exists a unique log stable curve C_s log étale over C_t such that the diagram

$$\begin{tikzcd} C'_s \arrow[r] \arrow[d] & C_s \arrow[d] \\ C'_t \arrow[r] & C_t \end{tikzcd}$$

commutes, where $C'_s \to C'_t$ is the log étale cover induced by $T' \to (\mathscr{M}_\eta^{\log}, L_\eta)$ and C_t is the log stable curve induced by $T \to (\overline{\mathscr{M}}_{r(\eta_t)}, L)$. Because $C'_t \to C_t$ is an exact closed immersion and $C'_s \to C'_t$ is étale , the existence and the uniqueness immediately follow from [38, Theorem 5.6]. □

4.3 Tropicalizing the Moduli Space $\mathscr{M}_\eta^{\log}$ of Logarithmic Étale Covers

In this section, we will review the relations between the cone stack $\mathscr{M}_{g,n}^{\mathrm{trop}}$ and $\mathscr{M}_{g,n}^{\log}$ in [9, Chapter 6] and apply the method there to get similar results for $\mathscr{M}_\eta^{\log}$.

Log geometry is closely related to tropical geometry.

Given a log scheme X, *the tropicalization via log structure* $\mathrm{Trop}(X)$ of X is

$$\mathrm{Trop}(X) := (\coprod_{x\in \underline{X}} \mathrm{Hom}(\overline{M}_{X,x}, \mathbb{R}_{\geq 0}))/\sim,$$

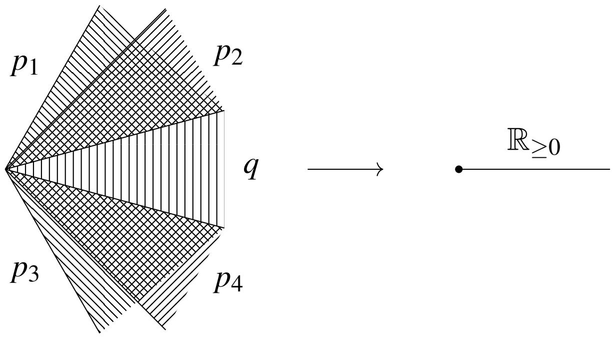
where the x runs over all scheme-theoretic points of $\underline{X}$ and the equivalence relation is given by dualizing the generization maps $\overline{M}_{X,x} \to \overline{M}_{X,x'}$ when x is specialization of x'.

The tropicalization via log structure is covariant. Indeed, if there is log morphism $f : X \to Y$, then there are morphisms $M_{Y,f(x)} \to M_{X,x}$ for $x \in X$ which are compatible with the equivalence relation. Hence we have the morphism $\mathrm{Trop}(X) \to \mathrm{Trop}(Y)$.

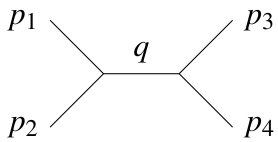
Example 4.3.1. [2, Example 2.1.6] Assume the log scheme X is the toric variety associated to the fan Σ equipped with the toric log structure. Then the tropicalization via log structure of X is precisely Σ.

Example 4.3.2. This example is the tropicalization via log structure of a log stable curve. Let C be the genus 0 log stable curve with 4 marked points which has two irreducible components with 2 marked points on each component over $(\mathrm{Spec}\, k, \mathbb{N})$. Assume at the only nodes, the log structure is determined by $\mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2$ with $\mathbb{N} \to \mathbb{N}, 1 \mapsto d$.

The tropicalization of C can be visualized as gluing 5 pieces of $\mathbb{R}^2$ corresponding to the 4 marked points p_1, p_2, p_3, p_4 and the nodes q. Assume p_1 and p_2 are on the same component while p_3 and p_4 are on the other. Then the pieces of p_1 and p_2 are glued along one axis of the piece of q while the pieces of p_3 and p_4 are glued along the other axis of the piece of q.



On the right-hand side is the tropicalization of $(\operatorname{Spec} k, \mathbb{N})$. The map is the induced map from tropicalization. Note that the inverse image of $1 \in \mathbb{R}_{\geq 0}$ is precisely the dual intersection graph of C. Moreover, the length of the bounded edge actually reflects the parameter d defining the log structure. This is how tropicalization via log structure is related to dual intersection graph.



Suppose that $\mathscr{C}$ is a category fibered in groupoids over **RPC**. There is an associated category fibered in groupoids $\mathscr{A}_{\mathscr{C}}$ over the category of logarithmic schemes by setting

$$\mathscr{A}_{\mathscr{C}}(S) = \mathscr{C}(\Gamma(S, \overline{M}_S)^{\vee})$$

for a logarithmic scheme S. Note that a category fibered in groupoids over **RPC** can be extended to a functor defined on all monoids [9, Section 5.2].

In general the category $\mathscr{A}_{\mathscr{C}}$ is not a stack. However, if $\mathscr{C}$ is a rational polyhedral cone σ, then $\mathscr{A}_{\mathscr{C}}$ is described below.

Proposition 4.3.3. *[37, Proposition 5.17] The stack $\mathscr{A}_\sigma$ is isomorphic to $[X_\sigma/T]$ where X_σ is the log scheme associated to σ. In particular, the stack $\mathscr{A}_\sigma$ is representable by an algebraic stack with a logarithmic structure.*

The stackification of $\mathscr{A}_{\mathscr{C}}$ is denoted by $\mathfrak{a}^*\mathscr{C}$. The reason for this notation is that the assignment $\sigma \mapsto \mathscr{A}_\sigma$ determines a morphism of sites $\mathfrak{a} : \text{Ét}(\mathbf{LSch}) \to \mathbf{RPC}_{\mathbb{Z}}$. See [9, Remark 6.10] for more details.

Definition 4.3.4. [9, Definition 6.8] An Artin cone is an algebraic stack with logarithmic structure that can be represented as the quotient $\mathscr{A}_\sigma$ for some rational polyhedral cone. An Artin fan is a logarithmic algebraic stack that has a strict étale cover by a disjoint union of Artin cones.

Theorem 4.3.5. *[9, Theorem 6.15] The functor $\mathfrak{a}^*$ defines an equivalence between the 2-category of cone stacks and the 2-category of Artin fans.*

Let $\mathfrak{a}^*\mathscr{M}_\eta^{\text{trop}}$ be the Artin fan of $\mathscr{M}_\eta^{\text{trop}}$.

Definition 4.3.6. Let X be a logarithmic scheme that is locally of finite type. A *tropical étale cover* over X consists of

(i) a tropical étale cover Γ_x with edge lengths in $\overline{M}_{X,x}$ (see Section 3.1) for each geometric point x of X;

(ii) if $y \rightsquigarrow x$ is a geometric specialization, then Γ_y is the tropical étale cover obtained from Γ_x by having the edge length

$$E(\mathbb{G}(\Gamma_{xt})) \xrightarrow{d_{\Gamma_x}} \overline{M}_{X,x} \to \overline{M}_{X,y}$$

and contracting those edges with length 0. Here Γ_{xt} is the target twisted tropical curve of Γ_x.

A tropical étale cover over a log scheme X is simply assigning to each geometric point a tropical étale cover compatible with the generization map. This can be viewed as tropicalization a family of log étale covers via log structure but preserving the base X.

We write $\tilde{\mathscr{M}}_\eta^{\text{trop}}$ for the fibered category over logarithmic schemes of finite type whose fiber over S is the category of families of tropical étale covers over S where for each geometric point of S, the tropical étale cover over S is of type η.

Let η be a graph cover whose target has 1 vertex and no edges. Let $\mathscr{M}_\eta^{\text{trop}}$ be the cone stack of tropical étale covers of type η defined in Chapter 3.

There is a natural morphism

$$\Phi : \mathfrak{a}^*\mathscr{M}_\eta^{\text{trop}}(S) \to \tilde{\mathscr{M}}_\eta^{\text{trop}}(S).$$

To be more precise, the morphism Φ is specified by the map $\mathscr{M}_\eta^{\text{trop}}(\Gamma(S,\overline{M}_S)^\vee) \to \tilde{\mathscr{M}}_\eta^{\text{trop}}(S)$ where the tropical étale cover $(G, d : E(G_t) \to \Gamma(S,\overline{M}_S))$ maps to the tropical étale cover

over S whose tropical étale cover at geometric point x of S is obtained from $(G, d : E(G_t) \to \Gamma(S, \overline{M}_S))$ by having the edge length

$$E(G_t) \xrightarrow{d} \Gamma(S, \overline{M}_S) \to \overline{M}_{S,x}$$

and contracting those edges with length 0.

Proposition 4.3.7. *The morphism $\Phi(S)$ is an isomorphism for log schemes S locally of finite type.*

Proof. The proof of [9, Lemma 6.18] works word for word. □

The *tropicalization map*

$$\mathrm{trop}_\eta : (\mathscr{M}_\eta^{\log}, L_\eta) \to \mathfrak{a}^* \mathscr{M}_\eta^{\mathrm{trop}} \simeq \tilde{\mathscr{M}}_\eta^{\mathrm{trop}}$$

is given by sending the basic log étale cover to its dual intersection graph with edge length at the edge corresponding to the double point q being the $\alpha \in Q$ as in Theorem 4.1.11.

Theorem 4.3.8. *The tropicalization map*

$$trop_\eta : (\mathscr{M}_\eta^{log}, L_\eta) \to \mathfrak{a}^* \mathscr{M}_\eta^{trop}$$

is strict and smooth.

Proof. The stack $\mathfrak{a}^* \mathscr{M}_\eta^{\mathrm{trop}}$ is log étale over $\mathrm{Spec}\,\mathbb{k}$ with the trivial log structure [9, Lemma 6.13]. By Proposition 4.2.8, the log stack $(\mathscr{M}_\eta^{\log}, L_\eta)$ is log smooth over $\mathrm{Spec}\,\mathbb{k}$ with the trivial log structure. So $\mathrm{trop}_\eta : (\mathscr{M}_\eta^{\log}, L_\eta) \to \mathfrak{a}^* \mathscr{M}_\eta^{\mathrm{trop}}$ is log smooth.

For strictness, it suffices to prove that every diagram

$$\begin{tikzcd} (\underline{S}, M_S) \arrow[r] \arrow[d] & \mathscr{M}_\eta^{\log} \arrow[d] \\ (\underline{S}, M'_S) \arrow[r] \arrow[ur, dashed] & \mathfrak{a}^* \mathscr{M}_\eta^{\mathrm{trop}} \end{tikzcd}$$

of solid arrows there is a unique lift of the dotted arrow where the underlying morphism $\underline{S} \to \underline{S}$ is the identity.

In other words, assume there is a log étale cover $C_s \to C_t$ over (S, M_S), a tropical étale cover Γ' over (S, M'_S) such that the family of tropical étale covers associated to (S, M_S) is the family Γ induced from Γ' by the morphism of log structures $M'_S \to M_S$. It suffices to show that there is a unique log étale cover $C_s' \to C_t'$ over (S, M_S') with the same underlying scheme morphism as $C_s \to C_t$ inducing both $C_s \to C_t$ and Γ'.

It suffices to define the log structure on $\underline{C_t}$ because the log structure on $\underline{C_s}$ is then completely determined. By [9, Theorem 6.20], there is a unique log stable curve C_t' over (S, M_S') inducing both C_t and the weighted marked graph $r(\Gamma_s')$. The twisted degree on Γ_s', the twisted graph Γ_t' and the log structure on C_s' are then given by the underlying scheme morphism of $C_s \to C_t$. $\square$

Now we have showed the tropicalization of the log stack $(\mathscr{M}_\eta^{\log}, L_\eta)$ via log structure is the same as the Artin stack of $\mathscr{M}_\eta^{\text{trop}}$. Moreover, the map from $\mathscr{M}_\eta^{\log}$ to its tropicalization $\mathfrak{a}^* \mathscr{M}_\eta^{\text{trop}}$ is smooth and strict.

Future work would involve comparing $\mathscr{M}_\eta^{\log}$ and $\overline{\mathscr{M}}_\eta$. It is conjectured that the map $\overline{\mathscr{M}}_\eta \to \mathscr{M}_\eta^{\log}$ defined as mapping a family of étale covers to the family of coarse moduli space equipped with the basic log structure is étale . The automorphism group $\mathrm{Aut}(\mathscr{C}' \to \mathscr{C} | C' \to C)$ is

$$\oplus_{l \in L(\eta_t)} \mu_{m(l)}$$

where η_t is the type for twisted curve $\mathscr{C}$, i.e., the automorphism group splits as a product with the contribution from each twisted node being trivial and the contribution from each marked points being $\mu_{m(l)}$.

References

[1] Abramovich, D., Caporaso, L., and Payne, S. (2014a). The tropicalization of the moduli space of curves. *Ann. Sci. Éc. Norm. Supér.(4)*.

[2] Abramovich, D., Chen, Q., Gross, M., and Siebert, B. (2013). Decomposition of degenerate Gromov–Witten invariants.

[3] Abramovich, D., Chen, Q., Marcus, S., and Wise, J. (2014b). Boundedness of the space of stable logarithmic maps. *arXiv preprint arXiv:1408.0869*.

[4] Abramovich, D., Corti, A., and Vistoli, A. (2003). Twisted bundles and admissible covers. *Communications in Algebra*, 31(8):3547–3618.

[5] Abramovich, D. and Vistoli, A. (2002). Compactifying the space of stable maps. *Journal of the American Mathematical Society*, 15(1):27–75.

[6] Abramovich, D. and Wise, J. (2013). Invariance in logarithmic Gromov-Witten theory. *arXiv preprint arXiv:1306.1222*.

[7] Allermann, L. and Rau, J. (2010). First steps in tropical intersection theory. *Mathematische zeitschrift*, 264(3):633–670.

[8] Blankers, V. (2017). Hyperelliptic classes are rigid and extremal in genus two. *arXiv preprint arXiv:1707.08676*.

[9] Cavalieri, R., Chan, M., Ulirsch, M., and Wise, J. (2017). A moduli stack of tropical curves. *arXiv preprint arXiv:1704.03806*.

[10] Cavalieri, R., Markwig, H., and Ranganathan, D. (2016). Tropicalizing the space of admissible covers. *Mathematische Annalen*, 364(3-4):1275–1313.

[11] Cavalieri, R. and Miles, E. (2015). From Riemann surfaces to algebraic geometry: A first course in Hurwitz theory. *Cambridge University Press*, 2:1.

[12] Costello, K. (2006). Higher genus Gromov-Witten invariants as genus zero invariants of symmetric products. *Annals of mathematics*, pages 561–601.

[13] Ekedahl, T., Lando, S., Shapiro, M., and Vainshtein, A. (2001). Hurwitz numbers and intersections on moduli spaces of curves. *Inventiones mathematicae*, 146(2):297–327.

[14] Fulton, W. (1993). *Introduction to toric varieties*. Princeton University Press.

[15] Fulton, W. and Pandharipande, R. (1996). Notes on stable maps and quantum cohomology. *arXiv preprint alg-geom/9608011*.

[16] Fulton, W. and Sturmfels, B. (1997). Intersection theory on toric varieties. *Topology*, 36(2):335–353.

[17] Gathmann, A. (2006). Tropical algebraic geometry. *Jahresber. Deutsch. Math.-Verein.*, 108(1):3–32.

[18] Gibney, A. and Maclagan, D. (2011). Equations for Chow and Hilbert quotients. *Algebra & Number Theory*, 4(7):855–885.

[19] Gillet, H. (1984). Intersection theory on algebraic stacks and q-varieties. *Journal of Pure and Applied Algebra*, 34(2-3):193–240.

[20] Gross, A. (2015). Intersection theory on tropicalizations of toroidal embeddings. *arXiv preprint arXiv:1510.04604*.

[21] Gross, M. (2011). *Tropical geometry and mirror symmetry*. American Mathematical Soc.

[22] Gross, M. and Siebert, B. (2003). Affine manifolds, log structures, and mirror symmetry. *Turk J Math*, 27:33–60.

[23] Gross, M. and Siebert, B. (2013). Logarithmic Gromov-Witten invariants. *Journal of the American Mathematical Society*, 26(2):451–510.

[24] Gross, M., Siebert, B., et al. (2006). Mirror symmetry via logarithmic degeneration data i. *Journal of Differential Geometry*, 72(2):169–338.

[25] Hartshorne, R. (1977). *Algebraic geometry*, volume 52. Springer.

[26] Hurwitz, A. (1891). Über Riemann'sche flächen mit gegebenen verzweigungspunkten. *Mathematische Annalen*, 39(1):1–60.

[27] Illusie, L. (1994). Logarithmic spaces (according to K. Kato). In *Barsotti symposium in algebraic geometry*, pages 183–203. Elsevier.

[28] Kato, F. (2000). Log smooth deformation and moduli of log smooth curves. *International Journal of Mathematics*, 11(02):215–232.

[29] Kato, K. (1989). Logarithmic structures of Fontaine-Illusie. *Algebraic Analysis, Geometry and Number Theory*, pages 191–224.

[30] Kerber, M. and Markwig, H. (2009). Intersecting psi-classes on tropical. *International Mathematics Research Notices*, 2009(2):221–240.

[31] Kontsevich, M. (1992). Intersection theory on the moduli space of curves and the matrix airy function. *Communications in Mathematical Physics*, 147(1):1–23.

[32] Markwig, H. and Rau, J. (2009). Tropical descendant Gromov–Witten invariants. *manuscripta mathematica*, 129(3):293–335.

[33] Mikhalkin, G. (2005). Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *Journal of the American Mathematical Society*, 18(2):313–377.

[34] Mikhalkin, G. (2006). Tropical geometry and its applications. In *Proceedings oh the International Congress of Mathematicians: Madrid, August 22-30, 2006: invited lectures*, pages 827–852.

[35] Mikhalkin, G. (2007). Moduli spaces of rational tropical curves. In *Proceedings of Gökova Geometry-Topology Conference 2006, 39–51, Gökova Geometry/Topology Conference (GGT).*

[36] Nishinou, T. and Siebert, B. (2006). Toric degenerations of toric varieties and tropical curves. *Duke Mathematical Journal*, 135(1):1–51.

[37] Olsson, M. C. (2003). Logarithmic geometry and algebraic stacks. In *Annales scientifiques de l'École normale supérieure*, volume 36, pages 747–791. Elsevier.

[38] Olsson, M. C. (2005). The logarithmic cotangent complex. *Mathematische Annalen*, 333(4):859–931.

[39] Ranganathan, D. (2017). Superabundant curves and the Artin fan. *Int. Math. Res. Not. IMRN*, (4):1103–1115.

[40] Serre, J. P. and Bass, H. (1977). *Arbres, amalgames, SL2: cours au Collège de France*. Société mathématique de France.

[41] Speyer, D. and Sturmfels, B. (2004). The tropical Grassmannian. *Advances in Geometry*, 4(3):389–411.

[42] Speyer, D. E. (2014). Parameterizing tropical curves I: Curves of genus zero and one. *Algebra Number Theory*, 8(4):963–998.

[43] Toën, B. and Vezzosi, G. (2008). *Homotopical Algebraic Geometry II: Geometric Stacks and Applications: Geometric Stacks and Applications*, volume 2. American Mathematical Soc.

[44] Witten, E. (1990). Two-dimensional gravity and intersection theory on moduli space. *Surveys in differential geometry*, 1(1):243–310.